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THE DYNAMICS OF A FLUCTUATING GROWTH RATE[1]

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Quantitative studies of growth phenomena should lead to a consideration of the energy relationships involved. We cannot but believe, in the absence of contrary evidence, that the life processes of living organisms are manifestations of the same principles of energy relationships which exist in other parts of the universe. The study of these energy relationships ought not, therefore, to be disregarded. Our aspirations for such knowledge, which have received so much encouragement from the discovery of the principle of the conservation of energy, can never be entirely suppressed.

This quantitative study of the rate of growth of shoots on a perennial plant, has been attempted in order to gain an insight into some kinetic aspects of the problem of growth. Growth, in one form or another, is the end product of most of the energy transformations of a tree. Not all forms of growth are readily measured. Increase in length is the most readily measured and its determination is fairly free from error. Increase in weight might be better, but where one wishes to study identical shoots through the season, weighing is impossible because it requires removal of the shoots from the tree.

The present study is based upon measurements of 70 selected shoots of Royal apricot trees which grew on the grounds of the Citrus Experiment Station at Riverside, California. The observations were made during the growing season of 1918. At the time the observations began, the trees had been growing two years in the orchard. The trees on which shoots were selected for measurement were scattered over the entire orchard in such a way that a fairly random distribution was obtained.

Shortly after the new shoots appeared, two were selected on each tree which had been pruned according to the "Winter-vase" type (a tree which is pruned every winter to produce a vase type). The type of pruning

was thus uniform, and since the trees were periodically irrigated throughout the season, it is believed that conditions necessary for fairly uniform growth were afforded. The trees, being so young, produced no fruit and consequently growth was not influenced by that activity. The selected shoots were marked with a line of India ink ten centimeters from the growing point and all measurements were made from this as a base line.

Most of the shoots produced several lateral branches during the season. The length of these laterals was disregarded as far as these studies are concerned. The propriety of this procedure may be questioned, yet it is difficult to see any valid objection thereto, because in measuring the increase in length of a growing shoot we are measuring the catalytic activity of a group of meristematic cells in the apical bud of that shoot. Unless the lateral branches of a shoot have some modifying effect either upon the growth catalyst, or upon the supply of material upon which the catalyst acts, it is difficult to see how the presence of the lateral shoots can modify the growth rate of the main branch. Since we know that removal of the lateral branches of a shoot retards rather than accelerates the length-growth of the main shoot, we are not warranted in assuming that their presence has a retarding influence upon the growth of the main shoot.

The Course of Growth of the Apricot Shoots.[2]—If the growth of a branch is similar in its rate to some form of autocatalytic reaction, we should expect the length of the branches to be determined by some sort of a unimolecular reaction formula which shall involve the time and a growth constant.

The most generally useful equation to express this sort of a rate is that of autocatalysis,

$$dx/dt = kx(a-x) \qquad (1)$$

in which a represents the final size of the plant, x is the height (or size) at any time, t, and k is the velocity constant of the reaction. A paper recently published (Reed and Holland, 1919) has shown that the growth rate of Helianthus is well expressed by this equation, but it does not necessarily follow that the growth rate of individual shoots on a perennial tree will follow the same type of growth as the axis of an annual plant like the sunflower.

Inspection of the observed values shows that there were three quite distinct cycles of growth in the growing season. The existence of these cycles may be seen by inspection of the graph in figure 1, but better by the observed increments in table 4. The large weekly increments in the first few weeks of the season diminish to a minimum value at the 9th week; then for a few weeks are larger, diminishing again to the 18th week; they increase to the 21st week and then decline to the end of the season. The season's growth, therefore, consisted of three cycles of growth each of about 9 weeks duration. These cycles were quite apparent during the

growing season. At the beginning of each cycle, growth was rapid, the leaves arising upon that part of the shoot were of maximum size and outspread. As the end of a growth cycle was approached, the slower-growing portion of the shoot produced smaller and smaller leaves which had more of a tendency to curl. Simultaneously, the apical bud of the shoot was enlarged and became almost completely dormant. Following the partial dormancy of the apical bud, lateral shoots were often formed from subapical buds. However, within a week or two, the next cycle of growth would start and the shoot would repeat the processes described for the preceding cycle.

If the shoots pass through distinct intra-seasonal cycles of growth, we

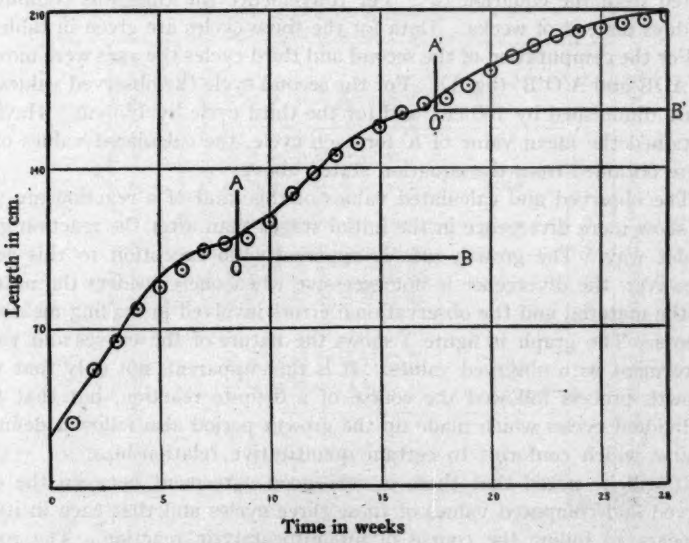

FIG. 1

Growth of apricot shoots represented as three cycles. ———, curve calculated from

$$\log \frac{x}{a - x} = k\,(t - t),$$

⊙ ⊙ ⊙ ⊙ , observed values.

should expect that the growth in each cycle should show evidence of being homogeneous and that it could be expressed definitely. We may, therefore, examine each of the three cycles of growth of the apricot shoots with this end in view. From inspection of the data it seems proper to designate the first cycle as the time from 0 to 8 weeks, the second from 9 to 17 weeks, and the third from 18 to 28 weeks. If each of these follows the course of a more or less independent autocatalytic reaction, we may compute their growth by the formula

$$\log \frac{x}{a-x} = K\,(t - t_1). \qquad (2)$$

which is the integral form of the equation (1), in which a is the length of the shoot at the end of the cycle, x is the length at any given time, t, $K = ak$, and t_1 is the time at which the cycle is half completed, i.e., when $x = a/2$.

The computations may be followed by the aid of table 1. Since the new shoots were marked at the commencement of observations at a uniform distance of 10 cm. from the tip, we may take the value of x as 10 when $t = 0$. The successive length of shoots in each cycle was computed from the equation (2). For convenience the time was computed in days instead of weeks. Data for the three cycles are given in table 1.

For the computation of the second and third cycles the axes were moved to AOB and A′O′B′ (fig. 1). For the second cycle the observed values of x are diminished by 100 cm. and for the third cycle by 165 cm. Having obtained the mean value of K for each cycle, the calculated values of x were obtained from the equation stated above.

The observed and calculated values of this kind of a reaction are apt to show more divergence in the initial stages than after the reaction gets under way. The growth rate is apparently no exception to this rule. However, the divergence is not excessive when one considers the nature of the material and the observational errors involved in making measurements. The graph in figure 1 shows the nature of the curves and their agreement with observed values. It is thus apparent, not only that the growth process followed the course of a definite reaction, but that the individual cycles which made up the growth period also follow a definite course which conforms to certain quantitative relationships.

It will be noted that there is very good agreement between the observed and computed values of these three cycles and that each in itself appears to follow the course of an autocatalytic reaction. The root-mean-square deviations of the values of the three cycles are 5.15 cm., 2.15 cm., and 2.52 cm.

It will be further noted that the mean value of K in the three cycles is successively less in each case, being, 0.0380, 0.0355, and 0.0277 respectively. Reference to the graph in figure 1 shows that the mean slope of the three curves is also successively less.

This analysis of the growth rate of the apricot shoots is taken to indicate that their seasonal growth was made up of three distinct cycles of approximately 9 weeks each. In the first cycle growth was more rapid than in any other. Growth goes on at a more rapid rate during approximately the first third of each cycle and then "fades away" toward the end of that cycle.

While this analysis affords some interesting insight into the growth

TABLE 1

GROWTH RATE OF APRICOT SHOOTS, CALCULATED UPON THE BASIS OF THREE CYCLES IN THE GROWING SEASON

1ST CYCLE					2ND CYCLE					3RD CYCLE				
t DAYS	x (OBSERVED) (CM.)	K	x (CALC.) (CM.)	θ (CM.)	t DAYS	x (OBSERVED) (CM.)	K	x (CALC.) (CM.)	θ	t DAYS	x (OBSERVED) (CM.)	K	x (CALC.)	θ (CM.)
0	10.0	0.0625	21.9	11.9	64	9.6	0.0419	12.2	2.6	126	5.3	0.0464	10.5	5.2
7	28.5*	0.0510	34.4	5.9	70	16.7	0.0418	19.0	2.3	133	11.8	0.0383	14.5	2.7
14	51.5	0.0295	50.2	−1.3	77	28.5	0.0320	27.8	−0.7	140	18.4*	0.0347	19.2	0.8
21	64.5	0.0294	67.1	2.6	84	37.4	0.0320	37.7	0.3	147	24.4	0.0285	24.4	0.0
28	78.6	0.0328	81.8	3.2	91	45.3*	0.0300	47.2	1.9	154	29.2	0.0273	29.3	0.1
35	88.0	0.0313	92.8	4.8	98	50.8	0.0269	54.9	4.1	161	33.0	0.0258	33.7	0.7
42	95.6	0.0310	100.1	4.5	105	58.4	0.0310	60.5	2.1	168	35.1	0.0224	37.2	2.1
49	103.9	0.0365	104.5	0.6	112	64.8	0.0374	64.2	−0.6	175	36.9	0.0206	39.9	3.0
56	107.2	0.0381	107.2	0.0	119	68.4	0.0463	66.5	−1.9	182	38.8	0.0201	41.9	3.1
										189	40.6	0.0203	43.2	2.6
										196	42.0	0.0205	44.1	2.1

Assumed value of a = 110.4
Assumed value of t_1 = 16.0
Mean value of K = 0.0380
Root-mean-square deviation of observed from calculated values of x = 5.15 cm.

Assumed value of a = 69.7
Assumed value of t_1 = 82.0
Mean value of K = 0.0355
Root-mean-square deviation of observed from calculated values of x = 2.15 cm.

Assumed value of a = 45.8
Assumed value of t_1 = 145.0
Mean value of K = 0.0277
Root-mean-square deviation of observed from calculated values of x = 2.52 cm.

* Interpolated.

rate of the apricot shoots it is not entirely satisfactory to dissect thus a process which is undoubtedly homogeneous in its nature. Our efforts must continually bend toward the problem of explaining the organism and its activities as a whole. We shall, therefore, attempt to study the dynamics of the entire season's growth and to learn something about the nature of the process.

Wilhelmy's equation, which has been found to apply to many processes in physical chemistry, was used;

$$\frac{dx}{dt} = k\,(a-x) \qquad (3)$$

TABLE 2

CALCULATED AND OBSERVED LENGTH OF APRICOT BRANCHES

TIME WEEKS	OBSERVED CM.	CALCULATED FROM $x=210\ (1-\epsilon^{-0.095t})$ CM.	θ
1	15.8	19.1	− 3.3
2	41.5	36.3	− 5.2
3	54.5	52.1	+ 2.4
4	68.6	66.4	+ 2.2
5	78.0	79.4	− 1.4
6	85.6	91.4	− 5.8
7	93.9	102.0	− 8.1
8	97.2	111.8	−14.6
9	99.6	120.8	−21.2
10	106.7	128.8	−22.1
11	118.5	136.0	−17.5
12	127.4	142.8	−15.4
13	135.3	148.8	−13.5
14	140.8	154.7	−13.9
15	148.4	159.5	−11.1
16	154.8	164.0	− 9.2
17	158.4	168.1	− 9.7
18	161.3	172.0	−10.7
19	166.8	175.6	− 8.8
20	173.4	178.8	− 5.4
21	179.4	181.2	− 1.8
22	184.2	184.0	0.2
23	188.0	186.5	1.5
24	190.1	188.6	1.5
25	191.9	190.4	1.5
26	193.8	192.2	1.6
27	195.6	193.8	1.8
28	197.0	195.2	1.8

where x represents the length of the shoot at time t, a represents the final length, and k is a constant of the reaction.

By integrating the above equation, we get

$$x = a\,(1-e^{-kt}) \qquad (4)$$

in which e is the base of the natural logarithms.

It is of interest to know whether the measurements of the apricot shoots will approximate values which will be given by the above equation. Reference to table 2, shows that the final mean length of the shoots was 197 cm. Let us assume that the limiting value of x is 210. Let this be

TABLE 3

LENGTH OF APRICOT SHOOTS AT WEEKLY INTERVALS DURING THE GROWING SEASON. VALUES OBSERVED AND CALCULATED FROM EQUATION (1)

t	x OBSERVED	$\{210[1-e^{-.095(t-1)}]\}_1$	$\{19.1\,e^{-.055t}\cos\frac{\pi}{14}t\}_2$	x CALCULATED $\{\}_1+\{\}_2$	θ
0	0	−19.1	19.1	0	0
1	15.8	0	17.6	17.6	−1.8
2	41.5	19.1	15.4	34.5	7.0
3	54.5	36.3	12.7	49.0	5.5
4	68.6	52.1	9.5	61.6	7.0
5	78.0	66.4	6.3	72.7	5.3
6	85.6	79.4	3.0	82.4	3.2
7	93.9	91.4	0	91.4	2.5
8	97.2	102.0	−2.7	99.3	−2.1
9	99.6	111.8	−5.0	106.8	−7.2
10	106.7	120.8	−6.8	114.0	−7.3
11	118.5	128.8	−8.1	120.7	−2.2
12	127.4	136.0	−8.9	127.1	0.3
13	135.3	142.8	−9.1	133.7	1.6
14	140.8	148.8	−8.8	140.8	0.8
15	148.4	154.7	−8.1	146.6	1.8
16	154.8	159.5	−7.1	152.4	2.4
17	158.4	164.0	−5.9	158.1	0.3
18	161.3	168.1	−4.4	163.7	−2.4
19	166.8	172.0	−2.9	169.1	−2.3
20	173.4	175.6	−1.4	174.2	−0.8
21	179.4	178.8	0	178.8	0.6
22	184.2	181.2	1.3	182.5	1.7
23	188.0	184.0	2.3	186.3	1.7
24	190.1	186.5	3.2	189.7	0.4
25	191.9	188.6	3.8	192.4	−0.4
26	193.8	190.4	4.1	194.5	−0.7
27	195.6	192.2	4.2	196.4	−0.8
28	197.0	193.8	4.1	197.9	−0.9

taken as the value of a. By a series of approximations it was found that the value of k is near 0.095. We may then write:

$$x = 210\ (1-e^{-.095t}) \qquad (5)$$

The values of x corresponding to values of t were determined and are given in table 2, together with the observed length of the shoots.

While there is a general harmony between the calculated and observed length of the branches at the weekly intervals, the values are not sufficiently close to be satisfactory. An examination of the figures shows that during the first seven weeks the computed length of the branches is fairly correct; from the 8th to the 20th week the computed values are too large; from the 20th week to the end of the season, the two sets of values agree fairly well.

The formula appears to fit the growth in the initial and latter part of the season, but growth lags behind the calculated values from the 8th to the 20th weeks.

This leads us to consider the possibility that the length-growth of these apricot branches follows the course of a reaction which consists of two consecutive unimolecular reactions, one of which at first accelerates and later retards the other. (The dynamics of such reactions have been discussed by Mellor, *Chemical Statics and Dynamics* (1914).) It is, therefore, necessary to increase the previously obtained values of x by some quantity which will be large when t is small and which will be comparatively small when t is large.

The nature of this quantity was ascertained by shifting all the values of t toward the ordinate, which was done by substituting $t-1$ for t in each case. The values by which the resulting determinations differed from the observed values were then computed and the differences were found to be periodically alternately positive and negative. (Compare columns 2 and 3 in table 3.) An appropriate form expressing these fluctuating differences may be a function of t involving the cosine. This will equal 1 when the value of the angle is 0, will decrease as the angle increases, then be negative as the angle increases from 90° to 270°, and become positive again in the last quadrant. By trial it was found that the difference between the observed values and those of the exponential function could be approximately expressed by

$$19.1\ [e^{-.055t}\cos\frac{\pi}{14}t]$$

The equation as used was then

$$x = 210\ [1-e^{.095\,(t-1)}] + 19.1\ [e^{-.055t}\cos\frac{\pi}{14}t] \qquad (6)$$

The values of x corresponding to the time intervals from 0 to 28 are given in the fifth column of table 3. The root-mean-square deviation

$(\Sigma d^2/n)^{1/2}$ of these results is 3.4 cm. The agreement, therefore, is satisfactory.

Comparing the calculated values of x with the observed, it will be seen that there are 16 deviations on the + side and 12 deviations on the − side. In figure 2, the upper curve shows the calculated course of the growth process alongside of the actual observed values.

The writer is fully aware that equation (6) is somewhat cumbersome. Its claim for consideration is, however, an important one, viz., it satisfactorily represents the observed facts. In the present state of our knowledge it seems advisable to use a formula like the above rather than a simpler type which might not so accurately represent the actual growth

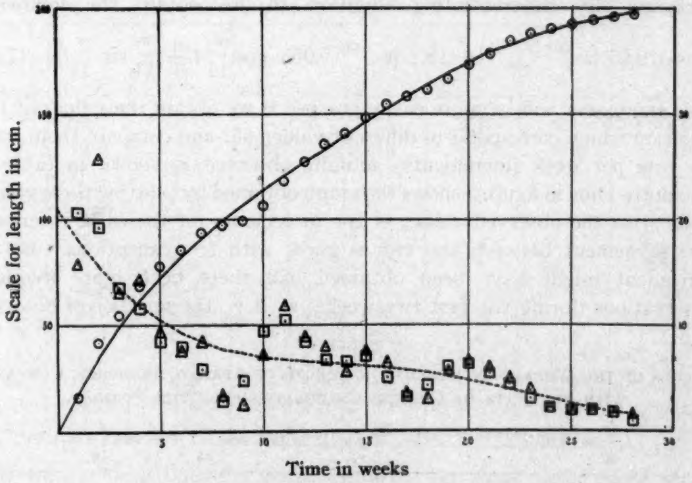

FIG. 2

Growth of apricot shoots. _____ curve calculated from

$$x = 210[1 - e^{-.095\,(t-1)}] + 19.1\,[e^{-.055t}\cos\tfrac{\pi}{14}t]$$

– – – – – – – – curve calculated from

$$\frac{dx}{dt} = 19.95\,e^{-.095\,(t-1)} - 19.1\,e^{-.055t}(0.055\cos\tfrac{\pi}{14}t + \tfrac{\pi}{14}\sin\tfrac{\pi}{14}t)$$

⊙ ⊙ ⊙, observed lengths of shoots.
△ △ △, observed increments of shoots.
⊡ ⊡ ⊡, "adjusted values" (S).

reaction. Future studies may show that a satisfactory equation of simpler form exists.

A consideration of equation (6) shows that we are here dealing with a homogeneous reaction, consisting of two unimolecular reactions, the first of which is always positive, and the second alternately positive and negative.

Putting this into terms of the tree, we may say that the growth of the apricot branches was, in the first 6 weeks, accelerated by amounts which diminished from 19.1 cm. at the outset to 3.0 cm. in the 6th week. From the 7th to the 21st week the growth was retarded by regularly varying amounts, and from then on, there was an acceleration, amounting at the most, to 4.2 cm.

The Dynamical Basis for Growth.—Our attention next passes to the question of the rate of growth. Since equation (6) represents the length of the apricot branches at successive intervals, we should obtain an equation by differentiating it, which will represent the rate at which they grow. Clearly, this relationship is of considerable physiological importance. By differentiating equation (6) we obtain the following

$$\frac{dx}{dt} = 19.95\,[e^{-.095\,(t-1)}] - 19.1\,[e^{-.055t}\,(.055\,\cos\frac{\pi}{14}t + \frac{\pi}{14}\sin\frac{\pi}{14}t)] \quad (7)$$

This expression will be more easily grasped if we obtain the values of the equation which correspond to different values of t and compare them with the rate per week (increments) actually observed as shown in table 4. The dotted line in figure 2 shows the graph obtained by plotting these values along with the observed values of the increments for the same intervals. The agreement between the two is good, with few exceptions. Better agreement might have been obtained had there been more frequent observations during the first two weeks; as it is, the number of observa-

TABLE 4

VALUES OF THE WEEKLY INCREMENTS IN LENGTH OF APRICOT BRANCHES, COMPARED WITH THE RATE OF GROWTH CALCULATED FROM THE FORMULA

$$\frac{dx}{dt} = 19.95[e^{-.095\,(t-1)}] - 19.1[e^{-.055t}(.055\cos\frac{\pi}{14}t + \frac{\pi}{14}\sin\frac{\pi}{14}t)]$$

t	OBSERVED CM.	CALCULATED CM.	θ	t	OBSERVED CM.	CALCULATED CM.	θ
0		20.9		15	7.6	6.1	1.5
1	15.8	18.1	−2.3	16	6.4	6.0	0.4
2	25.7	15.6	10.1	17	3.6	5.7	−2.1
3	13.0	13.5	−0.5	18	2.9	5.4	−2.5
4	14.1	11.8	2.3	19	5.5	5.1	0.4
5	9.4	10.4	−1.0	20	6.6	4.7	1.9
6	7.6	9.3	−1.7	21	6.0	4.3	1.7
7	8.3	8.4	−0.1	22	4.8	3.9	0.9
8	3.3	7.7	−4.4	23	3.8	3.4	0.4
9	2.4	7.3	−4.9	24	2.1	3.0	−0.9
10	7.1	6.9	0.2	25	1.8	2.5	−0.7
11	11.8	6.7	5.1	26	1.9	2.1	−0.2
12	8.9	6.6	2.3	27	1.8	1.7	0.1
13	7.9	6.4	1.5	28	1.4	1.5	−0.1
14	5.5	6.3	−0.8				

tions in the early part of the growth period are too scanty to give sufficient data for more accurate determinations. It will be noted that half of the observed values lie above the graph and half below it. This gives confidence in the reliability of the calculated values. The root-mean-square deviation of the 28 values is 2.78 cm.

Table 5 gives the values of S in comparison with the computed values of dx/dt. S is computed from

$^1/_2$ [observed length at time $(t + 1)$ − observed length at time $(t-1)$].

TABLE 5

VALUES OF S COMPUTED FROM $^1/_2$ [(OBS. LENGTH AT TIME $t + 1$) − (OBS. LENGTH AT TIME $t - 1$)] AND VALUES OF $\frac{dx}{dt}$.

t	S	$\frac{dx}{dt}$	θ	t	S	$\frac{dx}{dt}$	θ
1	20.7	18.1	−2.6	15	7.0	6.1	−0.9
2	19.3	15.6	−3.7	16	5.0	6.0	1.0
3	13.5	13.5	0.0	17	3.2	5.7	2.5
4	11.7	11.8	0.1	18	4.2	5.4	1.2
5	8.5	10.4	1.9	19	6.0	5.1	−0.9
6	8.0	9.3	1.3	20	6.3	4.7	−1.6
7	5.8	8.4	2.6	21	5.4	4.3	−1.1
8	2.8	7.7	4.9	22	4.3	3.9	−0.4
9	4.7	7.3	2.6	23	3.0	3.4	0.4
10	9.4	6.9	−2.5	24	2.0	3.0	1.0
11	10.3	6.7	−3.6	25	1.8	2.5	0.7
12	8.4	6.6	−1.8	26	1.8	2.1	0.3
13	6.7	6.4	−0.3	27	1.6	1.7	0.1
14	6.5	6.3	−0.2	28	0.7	1.5	0.8

If we use the values (S) which represent the slope of the chord whose mid-point corresponds to time, t, we shall obviate the minor fluctuations and get a more comparable expression of the observed rate of growth. McEwen and Michael (1919) have pointed out the usefulness of this method (p. 111) which depends upon the fact that the slope of the chord of a simple curve is approximately equal to that of the tangent at the point midway between the extremities of the chord. The root-mean-square deviations of these values is only 1.91 cm., which indicates a satisfactory agreement.

When these computations are translated into terms of the growing tree it appears that the growth rate in the early part of the season is high, but that it declines as the season advances. The rate of growth of the apricot branches indicates that the growth-impelling agency is at a maximum at the beginning of the season and that it diminishes as growth proceeds. In the first seven or eight weeks the growth rate declines more rapidly than in any other period. From the eighth to the nineteenth

week the rate is fairly constant, only diminishing about 2.5 cm., but from that time on, the decline is considerably more rapid. At the end of the growing season the rate is less than one-tenth of the initial rate.

From a biological point of view, our interest in any equation representing the growth of organisms is not confined to evolving equations which shall represent their size at any given time. If we know the rate at which an organism grows and can express it quantitatively, we are in a position to analyze the processes by which it increases in size.

To facilitate matters, we may write equation (6) in a general form

$$x = a\,[1-e^{-k_1(t-1)}] + be^{-k_2 t}\cos \alpha t.$$

When differentiated, this becomes

$$\frac{dx}{dt} = ak_1e^{-k_1(t-1)} - be^{-k_2 t}\{k_2 \cos \alpha t + \alpha \sin \alpha t\}$$

which is the general form corresponding to equation (7). This can also be written in the form

$$\frac{dx}{dt} = k_1\left[(a-x) + \frac{-be^{-k_2 t}\{(k_2-k_1)\cos \alpha t + \alpha \sin \alpha t\}}{k_1}\right]$$

For the sake of brevity let

$$Y = -be^{-k_2 t}\,[(k_2-k_1)\cos \alpha t + \alpha \sin \alpha t]$$

We may then write

$$\frac{dx}{dt} = k_1\,[(a-x) + Y/k_1]. \qquad (8)$$

In the case of the apricot data

$$Y = 19.1\,e^{-.055t}\left\{-.04\,\cos\frac{\pi}{14}t + \frac{\pi}{14}\sin\frac{\pi}{14}t\right\}.$$

The advantage of this form of expression lies in the fact that Y may be expressed in terms of x, since equation (7) may be written

$$Y = \frac{dx}{dt} - k_1\,(a-x).$$

It seems, therefore, that the growth of these apricot shoots in one season conforms to that of a unimolecular consecutive reaction, in which the main reaction is influenced by a secondary reaction product. The rate of the main reaction would follow the path of curve A in figure 3, if it were alone operative. The secondary reaction (represented by curve B) has an additive effect on the main reaction and gives the resultant course shown by curve C. In the first part of the season, B has a negative value, its effect, therefore, is to diminish A. In the latter part of the season, B has positive values and accordingly augments A.

The writer is inclined to adopt the suggestion of Robertson (1913) concerning the progressive increase or decrease in the amount of growth-catalyst during a cycle. If we assume that in this sort of a reaction the

amount of catalyst is in some way proportional to values of dx/dt, there would be in the present instance a progressive decrease during the growing season. If the initial excess of catalyst is unfavorable to growth then the effect of Y (which is negative during the first part of the season) would accelerate the reaction because it operates to cause a more rapid decrease in the amount of the catalyst. The amount of this acceleration is actually indicated by the values of 19.1 $[e^{-.055t} \cos \pi/14]$ in equation (6).

If the assumptions made in this study have any basis, it would seem that some light has been thrown upon the process of growth. It appears that growth proceeds at a definite rate, and is capable of mathematical expression. The growth rate appears to be proportional to the amount

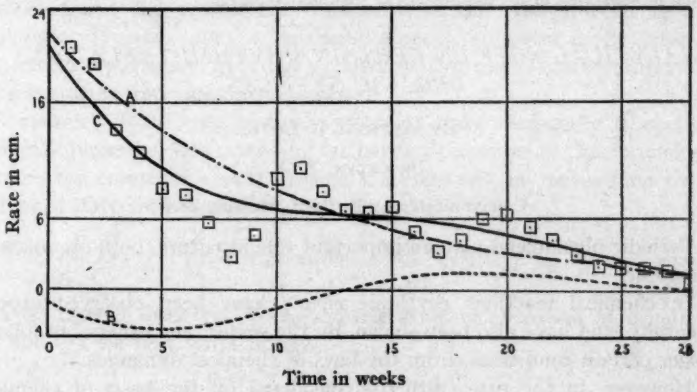

FIG. 3

Growth rate of apricot shoots.
—.—.—, (A), curve calculated from $ake^{-k_1}(t-1)$
———, (B), curve calculated from $-be^{-k_2 t}[k_2 \cos \alpha t + \alpha \sin \alpha t]$
———, (C), curve calculated from sum of values of (A) and (B).
⊡ ⊡ ⊡ observed increments.

of a hypothetical catalyst and the intra-seasonal cycles of growth may be indicative of fluctuations in the amount of active catalyst in the organism.

Summary.—1. The growth of shoots on a selected sample of young apricot trees, followed a definite, though fluctuating rate. The maximum growth rate was exhibited soon after the growth of the season began, and declined with some regularity to the end of the season, though three distinct intra-seasonal cycles of growth were apparent.

2. The growth in each cycle closely resembled the rate of an autocatalytic reaction. The growth rate for the entire season conforms to that of a reaction consisting of two unimolecular reactions, one of which at first accelerates and subsequently retards the other.

3. If growth be assumed to be proportional to the amount of active catalyst present, a method is available for studying the dynamics of the growth process.

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[1] Paper No. 64, University of California, Graduate School of Tropical Agriculture and Citrus Experiment Station, Riverside, California.

[2] The writer wishes to acknowledge his great indebtedness to Dr. G. F. McEwen of the Scripps Institution for Biological Research of the University of California for valuable assistance in the mathematical work here reported.

ANALYTICAL NOTE ON CERTAIN RHYTHMIC RELATIONS IN ORGANIC SYSTEMS

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Periodic phenomena play an important rôle in nature, both organic and inorganic.

In chemical reactions rhythmic effects have been observed experimentally, and have also been shown, by the writer[1] and others,[2] to follow, under certain conditions, from the laws of chemical dynamics.

However, in the cases hitherto considered on the basis of chemical dynamics, the oscillations were found to be of the damped kind, and therefore, only transitory (unlike certain experimentally observed periodic reactions). Furthermore, in a much more general investigation by the writer, covering the kinetics not only of chemical but also of biological systems, it appeared, from the nature of the solution obtained, improbable[3] that undamped, permanent oscillations would arise in the absence of geometrical, structural causes, in the very comprehensive class of systems considered. For it seemed that the occurrence of such permanent oscillations, the occurrence of purely imaginary exponents in the exponential series solution presented, would demand peculiar and very specific relations between the characteristic constants of the systems undergoing transformation; whereas in nature these constants would, presumably, stand in random relation.

It was, therefore, with considerable surprise that the writer, on applying his method to certain special cases, found these to lead to undamped, and hence indefinitely continued, oscillations.

As the matter presents several features of interest, and illustrates certain methods and principles, it appears worth while to set forth the argument and conclusions here.

Starting out first from a broad basis, we may consider a system in the process of evolution, such a system comprising a variety of species of matter $S_1, S_2 \ldots S_n$ of mass $X_1, X_2 \ldots X_n$. The species of matter S may be defined in any suitable way. Some of them may, for example, be biological species of organism, others may be components of the "inorganic environment." Or, the species of matter S may be several components of an inorganic system in the course of chemical transformation.

We may think of the state of the system at an instant of time as being defined by statement of the values of $X_1, X_2 \ldots X_n$; of certain parameters Q defining the character of each species (in general, variable with time); and of certain other parameters P. The parameters P will, in general, define the geometrical constraints of the system, both at the boundaries (volume, area, extension in space), and also in its interior (structure, topography, geography); they will further define such factors as temperature and climatic conditions.

For a very broad class of cases, including those commonly treated in chemical dynamics, but extending far beyond the scope of that branch of science, the course of events in such a system will be represented by a system of differential equations of the form

$$\left.\begin{aligned} \frac{dX_i}{dt} &= F_i(X_1, X_2 \ldots X_n; P, Q) \\ (i &= 1, 2 \ldots n) \end{aligned}\right\} \qquad (1)$$

If we restrict ourselves to the consideration of evolution at constant P's and Q's we may write briefly

$$\frac{dX_i}{dt} = F_i(X_1, X_2 \ldots X_n). \qquad (2)$$

The writer has elsewhere[4] given a somewhat detailed discussion of the general case, in which no special assumption is made regarding the form of the functions F, that is to say, regarding the mode of physical interdependence of the several species and their environment.

We now proceed to consider a simple special case, as follows:

The system comprises

1. A species of organism S_1, a plant species, say, deriving its nourishment from a source presented in such large excess that the mass of the source may be considered constant during the period of time with which we are concerned.

2. A species S_2, for example a herbivorous animal species, feeding on S_1.

In this case we have the following obvious relations

$$\left.\begin{array}{l}\text{Rate of increase of } X_1 \\ \text{per unit of time}\end{array}\right| = \left|\begin{array}{l}\text{Mass of newly} \\ \text{formed } S_1 \text{ per} \\ \text{unit of time}\end{array}\right| - \left|\begin{array}{l}\text{Mass of } S_1 \\ \text{destroyed by} \\ S_2 \text{ per unit of} \\ \text{time}\end{array}\right| - \left|\begin{array}{l}\text{Other dead} \\ \text{or excretory} \\ \text{matter elimi-} \\ \text{nated from } S_1 \\ \text{per unit of} \\ \text{time}\end{array}\right| \qquad (3)$$

$$\left\{\begin{array}{l}\text{Rate of increase of } X_2 \\ \text{per unit of time}\end{array}\right\} = \left\{\begin{array}{l}\text{Mass of newly formed } S_2^* \text{ per unit of time (derived from } S_1 \text{ as food ingested)}\end{array}\right\} - \left\{\begin{array}{l}\text{Mass of } S_2 \text{ destroyed or eliminated per unit of time}\end{array}\right\} \quad (4)$$

Or, in analytical symbols,

$$\frac{dX_1}{dt} = A'_1X_1 - B_1X_1X_2 - A''_1X_1 \quad (5)$$

$$= (A'_1 - A''_1)X_1 - B_1X_1X_2 \quad (6)$$

$$= A_1X_1 - B_1X_1X_2 \quad (7)$$

$$= X_1(A_1 - B_1X_2) \quad (8)$$

$$\frac{dX_2}{dt} = A_2X_1X_2 - B_2X_2 \quad (9)$$

$$= X_2(A_2X_1 - B_2) \quad (10)$$

The coefficients A_1, A_2, B_1, B_2 are in general functions of X_1 and X_2.

The reasons for selecting the form (5), (9) for the analytical formulation of (3), (4) require perhaps a little explanation. For small changes the rate of formation of new material of a given species of organism under given conditions is proportional to the existing mass of that species. In other words, the growth of living matter is a typically autocatakinetic[5] process. This term has, therefore, been put in the form $A' X_1$ for the species S_1. Proportionality does not hold for large changes of X_1, X_2, and this is duly provided for in that A'_1 is a function of X_1, X_2.

Similarly the mass of matter rejected per unit of time from the species S_1 is proportional to X_1, and has been put in the form A''_1X_1, where A'' is in general a function of X_1, X_2.

Again, the mass of S_1 destroyed by S_2 feeding upon it will, for small changes, be proportional to X_2 and also to X_1. This term has, therefore, been set down in the form $B_1X_1X_2$. Here again the departures from proportionality are taken care of by the variations of B_1 with X_1 and X_2, of which variables B_1 is a function.

Similar remarks apply to the formulation (9) of (4). If there were no waste in the feeding process, and assuming that S_2 consumes no other substance than S_1, we would have $B_1 = A_2$; but in the more general case $B_1 \neq A_2$. Approaching now the analytical treatment of equations (5), (9), or their equivalents (8), (10), we note first of all that there are two ways of satisfying the condition for equilibrium, namely:

$$X_1 = X_2 = 0 \quad (11)$$

and

$$X_1 = \frac{B_2}{A_2};\ X_2 = \frac{A_1}{B_1} \quad (12)$$

We shall return later to the condition (11).

Condition (12) we will employ to define a new origin. Accordingly we introduce into (8), (10) the variables:

$$x_1 = X_1 - \frac{B_2}{A_2} \tag{13}$$

$$x_2 = X_2 - \frac{A_1}{B_1} \tag{14}$$

and obtain

$$\begin{aligned} \frac{dx_1}{dt} &= a_{12}x_2 + a_{112}x_1x_2 \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{212}x_1x_2 \end{aligned} \tag{15}$$

where

$$a_{12} = -\frac{B_1B_2}{A_2} \tag{16}$$

$$a_{112} = -B_1 \tag{17}$$

$$a_{21} = \frac{A_1A_2}{B_1} \tag{18}$$

$$a_{212} = A_2 \tag{19}$$

Note the significant fact that in (15) the linear terms in the dexter diagonal are lacking. It is this circumstance which imparts an oscillatory character to the process.

For, since a_{12} and a_{21} are in general functions of x_1, x_2, let us expand them by Taylor's theorem and put

$$a_{12} = p_0 + p_1x_1 + p_2x_2 + \ldots. \tag{20}$$

$$a_{21} = q_0 + q_1x_1 + q_2x_2 + \ldots. \tag{21}$$

A general solution of the system of differential equations (15) is then

$$x_1 = P_1e^{\lambda_1 t} + P_2e^{\lambda_2 t} + P_{11}e^{2\lambda_1 t} + P_{22}e^{2\lambda_2 t} + \ldots. \tag{23}$$

$$x_2 = Q_1e^{\lambda_1 t} + Q_2e^{\lambda_2 t} + Q_{11}e^{2\lambda_1 t} + Q_{22}e^{2\lambda_2 t} + \ldots. \tag{24}$$

where λ_1, λ_2, are the roots of the determinental equation for λ

$$\begin{vmatrix} -\lambda & p_0 \\ q_0 & -\lambda \end{vmatrix} = 0 \tag{25}$$

that is to say,

$$\lambda = \pm\sqrt{p_0q_0} \tag{26}$$

Now, according to (20), (21) p_0, q_0 are the equilibrium values of a_{12}, a_{21}. Hence, if we denote by $\bar{A}_1$, $\bar{B}_2$ the equilibrium values of A_1, B_2, i.e., those values which correspond to $x_1 = x_2 = 0$, then we have, by (16), (18)

$$p_0q_0 = -\bar{A}_1\bar{B}_2 \tag{27}$$

and hence

$$\lambda = \pm\sqrt{-\bar{A}_1\bar{B}_2} \tag{28}$$

Now the coefficient B_2 is, in the nature of things, a positive number, as follows from its definition by (4), (9).

As regards the coefficient A_1, we have two possible alternatives.

If A_1 is negative for all values of X_1, X_2, then λ, as defined by (28), would be real; but this inference is nugatory. For B_1, like B_2 is, by definition (3), (5), an essentially positive quantity, and hence the equilibrium defined by (12) would in this case occur at a negative value of X_2. But this is physically impossible, since X_2 is a mass.

By referring to (5), (7) it will be seen that this case, in which A_1 is negative for all values of X_1, X_2, and in which an equilibrium of the type defined by (12) is physically impossible, corresponds to a species S_1 incapable of maintaining itself even in the absence of the tax placed upon it by the species S_2 feeding upon it. This is a case of minor interest.

If, on the contrary (12) can be satisfied by a positive value of A_1, so that an equilibrium of the type (12) is physically possible, then, evidently, by (28), λ is a pure imaginary. The solution (23), (24) then takes the form of Fourier's series; the process is an undamped oscillation continuing indefinitely.

In this connection, it is interesting to recall a passage in Spencer's "First Principles," chapter 22, paragraph 173:

"Every species of plant and animal is perpetually undergoing a rhythmical variation in number—now from abundance of food and absence of enemies rising above its average, and then by a consequent scarcity of food and abundance of enemies being depressed below its average..... .....amid these oscillations produced by their conflict, lies that average number of the species at which its expansive tendency is in equilibrium with surrounding repressive tendencies. Nor can it be questioned that this balancing of the preservative and destructive forces which we see going on in every race must necessarily go on. Since increase of numbers cannot but continue until increase of mortality stops it, and decrease of numbers cannot but continue until it is either arrested by fertility or extinguishes the race entirely."

A question now arises. Do the curves representing the solution (23), (24) dip below the zero axes of X_1, X_2? This would mean that one or the other, or both, of the species S_1, S_2 would become extinct through the violence of the oscillations.

To answer this question we consider the relation:

$$\frac{dX_2}{dX_1} = \frac{X_2(A_2X_1 - B_2)}{X_1(A_1 - B_1X_2)} \tag{29}$$

which is obtained from (8) and (10) by division. From the periodicity of x_1, x_2 (and, therefore, X_1, X_2) it follows that the curve defined in rectangular coördinates X_1, X_2 by (29) is a closed curve. Furthermore, this curve can never cross the X_1 axis, for at all points of this axis the first,

and all the higher derivatives of X_2 with regard to X_1 vanish, as can be seen from (29) directly and by successive differentiations.

Similarly it can be seen that the curve defined by (29) can never cross the X_2 axis.

Hence, if any point on any integral curve of (29) lies within the positive quadrant, the whole of that curve lies in that quadrant. Thus the oscillations can never exceed the limits of positive values X_1, X_2.

We conclude, therefore, that under the conditions of the problem as here set forth, neither the species S_1 nor the species S_2 can become extinct through severity of the oscillations alone. In practice the eventuality might arise, however, that in the course of these oscillations one or the other species might be so thinned out as to succumb to any extraneous influence that might arise such as has not been taken into account in our present considerations.

We return now briefly to the consideration of the equilibrium defined by the equation

$$X_1 = X_2 = 0 \qquad (11)$$

Applying here the criterion set forth by the author elsewhere,[6] it is seen that when A_1 is positive the determinental equation for λ has at this point two real roots of opposite sign, which is characteristic of unstable equilibrium. If, on the other hand, A_1 is negative in the neighborhood of the origin of X_1, X_2, then the equilibrium here is found to be stable, the two roots for λ being both negative.

In conclusion it may be remarked that a system of equations identical in form with (8), (10) is obtained in the discussion of certain consecutive autocatalytic chemical reactions. Here, however, the coefficients A, B are constants and the integration can be reduced to a quadrature. Aside from a certain number of periodic reactions which have been observed more or less as laboratory curiosities, a certain interest is also attached to this matter from the fact that rhythmical reactions (e.g., heartbeat, which may continue after excision), play an important rôle in physiology. We cannot, of course, say whether in such case geometrical (structural) features are the dominating factors.

[1] Lotka, A. J., *J. Phys. Chem.*, **14**, 1910 (271–274); *Zs. physik. Chem.*, **72**, 1910 (508–511); **80**, 1912 (159–164); *Phys. Rev.*, **24**, 1912 (235–238); *Proc. Amer. Acad. Arts Sci.*, **55**, 1920 (137–153).

[2] Hirniak, J., *Zs. physik. Chem.*, **75**, 1910 (675); compare also Lowry and John, *J. Chem. Soc.*, **97**, 1910 (2634–2645).

[3] Lotka, A. J., *Proc. Amer. Acad.*, loc. cit., p. 145, footnote 13.

[4] Lotka, A. J., *Science Progress*, **14**, 1920 (406–417); *Proc. Amer. Acad.*, loc. cit.

[5] Ostwald, Wo., *Über die zeitlichen Eigenschaften der Entwickelungsvorgänge*, Leipsic, 1908, p. 36.

[6] Lotka, A. J., *Proc. Amer. Acad.*, loc. cit., p. 144, et seq.

ON THE CLASS NUMBER OF THE FIELD $\Omega(e^{2i\pi/p^n})$ AND THE SECOND CASE OF FERMAT'S LAST THEOREM

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In the July 1919 number of the *Bulletin of the American Mathematical Society*, page 458, I gave an expression for the residue of the first factor of the class number of the field defined by $e^{2i\pi/p}$, p being prime, with respect to the modulus p^a, in terms of Bernoulli numbers. In the present paper an analogous expression for the residue of the first factor of the class number of $\Omega(e^{2i\pi/p^n})$ modulo p, will be obtained and the result used to show that certain results due to Bernstein[1] on Fermat's last theorem do not have the generality stated by him. In view of the criticisms on Kummer's 1857 memoir on the last theorem which I have given elsewhere[2] it is then pointed out that up to the present time no rigorous proof of the theorem for the exponents 59 and 67 has been given.

Westlund[3] reduced the first factor of the class number of $\Omega(e^{2i\pi/p^n})$, if $n>1$, to the form

$$h_1 = k \times \frac{\prod_s \varphi(\theta^s)}{{}^1/_2 p^{n-2}(p-1)^2 \ {}^1/_2 np^{n-2}(p-1)^2-1} = kk_1 \quad (1)$$

where k is the first factor of the class number of $\Omega(e^{2i\pi/m^1})$, $\theta = e^{2i\pi/\mu}$, $\mu = \varphi(m) = p^{n-1}(p-1)$, $m = p^n$, $m^1 = p^{n-1}$, r_i is the least positive residue of r^i, modulo p^n, r being a primitive root of p^n. The integer s takes on all odd values $< \mu$ except multiples of p, and the function $\varphi(\theta)$ defined by Westlund may be put in the form

$$\varphi(\theta) = r_0 + r_1\theta + \ldots\ldots + r_{\mu-1}\theta^{\mu-1}.$$

We shall now reduce k_1 modulo p. To do this a modification of the method used by Kronecker in reducing the first factor of the class number of $\Omega(e^{2i\pi/p})$ will be employed.[4] We have

$$(r-\theta^{-1})\varphi(\theta) = p^n g(\theta) \quad (2)$$

where

$$g(\theta) = q_0 + q_1\theta + \ldots\ldots + q_{\mu-1}\theta^{\mu-1}$$

and

$$q_i = \frac{rr_i - r_{i+1}}{p^n}.$$

From an argument used by Westlund we also have

$$\prod_s (r-\theta^{-s}) = \frac{r^{\frac{\mu}{2}}+1}{r^{\frac{\mu^1}{2}}+1},$$

where $\mu^1 = \varphi(m^1)$. Since r is a primitive root of p^n, then $r^{\frac{\mu^1}{2}} = -1 +$

$p^{n-1}a$, where a is prime to p. Hence $r^{\frac{\mu}{2}} = (-1 + p^{n-1}a)^{\frac{\mu}{2}} = -1 + p^n a + p^{2n-1}a_1$, where a_1 is a multiple $a_2 a$ of a. Thus

$$\prod_s (r - \theta^s) = \frac{p^n a + p^{2n-1} a_1}{p^{n-1} a} = p + p^n a_2,$$

whence

$$(1 + a_2 p^{n-1}) \frac{\prod_s \varphi(\theta^s)}{{}^1/_2 n p^{n-2}(p-1)^2 - 1} \equiv \prod_s g(\theta^s)$$

and since

$$2^{1/2} p^{n-2}(p-1)^2 \equiv 1 \pmod{p},$$

we have

$$k_1 \equiv \prod_s g(\theta^s) \pmod{p}. \tag{3}$$

Now $\prod_s g(\theta^s) = a$, where a is a rational integer. Consider the expression $\prod_s g(x^s) - a$, where x is an indeterminate. This is a polynomial in x which vanishes for $x = \theta$, and hence is divisible by $V(x) = \prod_i (x - \theta^i)$, i ranging over the $\varphi(\mu)$ integers less than and prime to μ, since $V(x)$ is irreducible in the domain of rational integers. We write then

$$\prod_s g(x^s) = a + V(x)W(x), \tag{4}$$

where $W(x)$ is a rational integral function of x. Let

$$V_s = V_s(x) = \prod_i (x - \omega^i)$$

where i ranges over all the integers less than s and prime to it, and ω is a primitive s^{th} root of unity. Then

$$\frac{x^\mu - 1}{x^{\mu'} - 1} = V_\mu V_{c_1} V_{c_2} \ldots\ldots, \tag{5}$$

where $c_1, c_2 \ldots\ldots$ are all the numbers of the form $p^{n-1}k$, where k is a divisor of $p - 1$ which is $< p - 1$. Since r is a primitive root of p^n it is also a primitive root of p. Also, (5) gives

$$\frac{r^\mu - 1}{r^{\mu'} - 1} = V_\mu(r) V_{c_1}(r) V_{c_2}(r) \ldots\ldots \tag{6}$$

Now no V_c can be divisible by p since it would then follow that

$$r^c \equiv 1 \pmod{p}$$

which is impossible since c is not a multiple of $p - 1$. Since the left-hand member of (6) is divisible by p, we ,therefore, conclude that $V_\mu(r) \equiv 0 \pmod{p}$. Substituting $x = r$ in (4) and using (3), we have

$$k_1 \equiv \prod_s g(r^s) \pmod{p}. \tag{7}$$

By Fermat's theorem we have

$$\prod_s g(r^s) \equiv \prod_t (g(r^{2t-1}))^{p^n - 2(p-1)} \pmod{p}, \tag{7a}$$

where $t = 1, 2, \ldots\ldots, \frac{p-1}{2}$. To reduce $g(r^{2t-1})$ modulo p, we have

$$rr_i = r_{i+1} + rr_i - r_{i+1}$$

Raising to the power $2t$, and noting that $rr_i - r_{i+1} \equiv 0 \pmod{p^n}$, we get

$$r^{2t} r_i^{2t} \equiv r_{i+1}^{2t} + 2t(rr_i - r_{i+1}) r_{i+1}^{2t-1} \pmod{p^{2n}}.$$

Transposing, we have

$$2t r_{i+1}^{2t-1}(rr_i - r_{i+1}) \equiv r^{2t} r_i^{2t} - r_{i+1}^{2t} \pmod{p^{2n}}.$$

But, by definition,

$$r_{i+1} \equiv r^{i+1} \pmod{p^n}$$

and therefore

$$(rr_i - r_{i+1}) r_{i+1}^{2t+1} \equiv (rr_i - r_{i+1}) r^{(i+1)(2t-1)} \pmod{p^{2n}},$$

whence, if $i = 0, 1, \ldots\ldots, \mu - 1$,

$$2t p_n r^{2t-1} \sum_i q_i r^{i(2t-1)} \equiv r^{2t} \sum_i r_i^{2t} - \sum_i r_{i+1}^{2t} \pmod{p^{2n}}.$$

Now

$$\Sigma r_i^{2t} = \Sigma r_{i+1}^{2t} = a_1^{2t} + a_2^{2t} + \ldots\ldots + a_\mu^{2t} = S_{2t},$$

where the a's are the integers less than p^n and prime to it. Hence

$$\prod_t g(r^{2t-1}) \equiv \prod_t \frac{(r^{2t}-1) S_{2t}}{2t p^n r^{2t-1}} \pmod{p}. \tag{8}$$

To reduce the expression on the right-hand side we shall first consider the quantity S_{2t}. It will be shown that

$$S_{2t} \equiv R_{2t} \pmod{p^{n+1}} \tag{9}$$

for $t < (p-1)/2$ and

$$S_{2t} \equiv R_{2t} \pmod{p^n} \tag{9a}$$

for $t = (p-1)/2$, where

$$R_{2t} = 1^{2t} + 2^{2t} + \ldots\ldots + (p^n - 1)^{2t}.$$

We have

$$\begin{aligned} S_{2t} &= R_{2t} - (p^{2t} + (2p)^{2t} + (3p)^{2t} + \ldots\ldots + (p^n - p)^{2t}) \\ &= R_{2t} - p^{2t}(1^{2t} + 2^{2t} + \ldots\ldots + (p^{n-1} - 1)^{2t}). \end{aligned} \tag{9b}$$

Now

$$T_{2t} = 1^{2t} + 2^{2t} + \ldots\ldots + (p^{n-1} - 1)^{2t} \equiv 0 \pmod{p^{n-1}}$$

for $t < (p-1)/2$ and

$$T_{p-1} \equiv 0 \pmod{p^{n-2}}.$$

To show this let k be a primitive root of p^{n-1}, then $k^{2t} T_{2t} \equiv T_{2t} \pmod{p^{n-1}}$ and

$$(k^{2t} - 1) T_{2t} \equiv 0 \pmod{p^{n-1}}.$$

In this relation $k^{2t} - 1$ is prime to p for $t < (p-1)/2$ and for $2t = p - 1$, $k^{2t} - 1$ is divisible by p but not by p^2. Hence

$$T_{2t} \equiv 0 \pmod{p^{n-1}}$$

for $2t < p-1$ and

$$T_{p-1} \equiv 0 \pmod{p^{n-2}}. \qquad (9c)$$

Applying these to (9b), the relations (9) and (9a) follow. We also have by the Bernoulli summation formula

$$R_{2t} = \frac{p^{n(2t+1)}}{2t+1} + \frac{p^{2tn}}{2} + \binom{2t}{1}\frac{B_1}{2}p^{n(2t-1)} - \binom{2t}{3}\frac{B_2}{4}p^{n(2t-3)} + \ldots\ldots + (-1)^{t+1}B_t p^n.$$

where $B_1 = 1/6$, $B_2 = 1/30$, etc.

By the von Staudt-Clausen theorem, the Bernoulli numbers B_1, B_2, $\ldots\ldots$, $B_{(p-3)/2}$ all have denominators prime to p. Hence if $2t < p-1$ every term in the above expansion is an integer, or else a fraction whose denominator is prime to p. Also the numerators except in the last term, are divisible by p^{2n}. If $2t = p-1$, then the denominator of B_{p-1} is divisible by p but not by p^2 and, therefore, we may write

$$R_{p-1} \equiv p^{n-1}a \pmod{p^{2n}} \ (a \text{ an integer prime to } p). \qquad (10)$$

Hence we may write

$$R_{2t} \equiv (-1)^{t+1}B_t p^n \pmod{p^{n+1}}$$

and, therefore, for any $t < (p-1)/2$ we have

$$\frac{(r^{2t}-1)S_{2t}}{2tp^n r^{2t-1}} \equiv \frac{(r^{2t}-1)R_{2t}}{2tp^n r^{2t-1}} \equiv \frac{(-1)^{t+1}B_t(r^{2t}-1)}{2tr^{2t-1}} \pmod{p} \qquad (11)$$

For $t = (p-1)/2$, the corresponding factor in the right-hand member of (8) takes the form

$$\frac{(r^{p-1}-1)S_{p-1}}{(p-1)p^n r^{p-2}}. \qquad (11a)$$

We have from (9a)

$$S_{p-1} \equiv R_{p-1} \pmod{p^n}$$

and, therefore, the relation (10) shows that $(r^{p-1}-1)S_{p-1}$ is divisible by p^n but not by p^{n+1} since $r^{p-1}-1$ is divisible by p but not by p^2. Then modulo p, the expression (11a) may be reduced to an integer prime to p. Using this in connection with (11), (7a) and (8), we obtain

$$\prod_s g(r^s) \equiv \left(\prod_{t=1}^{(p-3)/2} B_t\right)^{p^{n-2}(p-1)} \pmod{p}. \qquad (12)$$

Hence, $\prod_s g(r^s)$ and, therefore, k, is divisible by p if and only if at least one of the numbers B_t ($t = 1, 2, \ldots\ldots (p-3)/2$) is divisible by p. (A Bernoulli number is said to be divisible by an integer i when its denominator is prime to i and its numerator is divisible by i.) Now Bernstein in his article cited above gives the result:

Under the assumption that the class number of $\Omega(e^{2i\pi/p^2})$ *is divisible by* p *but not by* p^2, *the relation*

$$x^p + y^p + z^p = 0 \qquad (13)$$

is impossible in integers prime to each other if $xyz \equiv 0 \pmod{p}$.

I shall now show that this criterion constitutes no extension over the one given by Kummer to the effect that (13) is impossible in integers if p is a regular prime. For if the former applies to other than regular primes as exponents, then p must be a prime such that

$$\prod_{i=1}^{(p-3)/2} B_i \equiv 0 \pmod{p}.$$

This being the case, it follows from (12) that $k_1 \equiv 0 \pmod{p}$, and by Kummer's theorem,[5] k is also divisible by p. Hence $h_1 \equiv 0 \pmod{p^2}$ contrary to Bernstein's assumption that the class number of $\Omega(e^{2i\pi/p^2})$ is not divisible by p^2.

In a paper cited above the writer pointed out errors in Kummer's memoir of 1857 on the relation (13). Considering this in connection with the failure of Bernstein's criterion it follows that Fermat's last theorem has not been rigorously proved for all non-regular primes less than 100. These non-regular primes are 37, 59 and 67. Mirimanoff[6] gave an adequate proof for the case $p = 37$, which leaves the cases $p = 59$ and 67 still to be disposed of.

In connection with the cyclotomic class number it may be noted that Furtwängler[7] proved that the class number of $\Omega(e^{2i\pi/p^n})$ is divisible by the prime p if, and only if, the first factor of the class number of the field $\Omega(e^{2i\pi/p})$ is divisible by p.

2. Bernstein[8] also gave a criterion in connection with (13) for the case where x, y and z are prime to p. He makes two assumptions the first being to the effect that the *second* factor of the class number of $\Omega(e^{2i\pi/p})$ is divisible by p, and states *l. c.*, p. 507, that his criterion includes that of Kummer's 1857 memoir as a special case. This is not correct, however, as there are no primes, p less than 100 such that the 2nd factor of the class number is divisible by p. (This follows from some computations by Kummer[9] in connection with certain investigations regarding the last theorem.) This being the case, Bernstein's criterion can at most supplement Kummer's which latter criterion stated that if (13) is satisfied in integers such that xyz is prime to p then $B_{(p-3)/3}$ and $B_{(p-5)/2}$ are divisible[10] by p.

This criterion would eliminate, for example, all p's less than 100, which Bernstein's condition does not do.

3. Kummer[11] rigorously established the theorem that (13) cannot be satisfied in integers prime to each other if p is a regular prime, and noted that all primes < 100 were regular except 37, 59 and 67. But Kummer[12] also showed that the first factor of the class numbers of $\Omega(e^{2i\pi/p})$ for all primes p, where $100 < p < 167$, excepting $p = 101, 103, 131, 149$ and 157, was in each case prime to p.[13] It follows that the primes p which are

> 100 and < 167 with the exceptions noted above are all regular and the last theorem is proved for these exponents p.

[1] *Göttingen Nachrichten*, 1910 (507–16).
[2] These PROCEEDINGS, **6**, 1920 (266).
[3] *Trans. Amer. Math. Soc.*, **4**, 1903 (201–12).
[4] Hilbert, *Die Theorie der Algebraischen Zahlkörpers*, p. 430
[5] Hilbert, *l. c.*, p. 429.
[6] *Crelle*, **111**, 1893 (26–30).
[7] *Ibid.*, **140**, 1911 (29).
[8] *L. c.*, pp. 482–88.
[9] *Abhandlungen Berlin Academy*, 1857, p. 73, verification of 2nd assumption.
[10] Kummer, *l. c.*, pp. 63–5.
[11] *Crelle*, **40** (93–139).
[12] *Berlin Monatsberichte*, 1874.
[13] Hilbert, *l. c.*, pp. 429, 435.

OBSERVATIONS ON THE STERILITY OF MUTANT HYBRIDS IN DROSOPHILA VIRILIS

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Communicated by C. B. Davenport, May 28, 1920

In an earlier paper[1] by Metz and Bridges, attention was called to two cases of apparent incompatibility between mutant races in Drosophila. One of these cases involved my data on the sex linked characters rugose and glazed in Drosophila virilis. Subsequent to the publication of this paper another character appeared in D. virilis that shows the same relations to rugose and glazed that they do to one another.[2] A study of these three characters has brought out some interesting relationships and has apparently revealed an error in the previous publication that should be corrected.

It may be recalled that hybrids between rugose and glazed were invariably sterile, while supposedly pure stock of each mutant was fertile and hybrids of either with other mutant stocks were fertile; hence it was concluded that rugose and glazed were incompatible. It appears now that a mistake was probably made in the statement that pure stocks of both rugose and glazed have normal fertility. Apparently this should have applied only to rugose, for later work has shown that females homozygous for glazed are usually sterile and that pure cultures cannot be maintained—at least without great difficulty. Possibly this condition has arisen by the secondary appearance of sterility factors in the stock since the previous paper was written, but more probably the earlier stock was impure, at least part of the time—the impurity having been overlooked through confusion with rugose in which the homozygous females are somatically normal. Unfortunately no accurate records of stock

bottles were kept, so the point cannot be definitely settled; but in any event the present degree of sterility in glazed stock suggests an interpretation of the sterility of the hybrids which differs considerably from that advanced previously. This is emphasized even more by the behavior of the third character, wax, as will appear below.

The three (allelomorphic?)[2] mutants, rugose, glazed and wax, form a graded series in respect to their morphological characteristics and also in respect to fertility, but contrary to expectation the order of dominance of the somatic manifestations appears to be just the reverse of that of fertility. These features may be brought out more clearly by a brief summary of the salient points. I am indebted to Dr. Alexander Weinstein and Miss Eleanor D. Mason for obtaining many of the records involved here.

1. *Rugose;* characterized somatically by a slight paling and a roughening of the eye, evident only in the male, the female being entirely normal in appearance; in fertility both sexes seem to be fully equal to the wild type stock, as determined by a large number of matings and by the behavior of the pure stock cultures.

2. *Glazed;* more extreme in all respects; the eyes have a glazed appearance in both sexes, although the males are more noticeably affected; as noted above the females are usually sterile; among approximately 150 females tested[3] 3 were fertile, giving respectively 6, 7 and 8 offspring;[4] apparently the males are also affected somewhat, for they do not breed as well as do rugose males.

3. *Wax;* still more extreme; the eyes of both sexes are greatly affected, resembling masses of yellow wax; the females seem to be practically or entirely sterile; among approximately 100 tested no certain case of fertility was found. One bottle gave a single normal female, another gave a normal female and wax male. The first case is probably due to contamination, and the latter may be accounted for in the same way, although the presence of the wax male adds to the probability of actual, but very low, fertility here. The males, in turn, appear to breed more poorly than do the glazed males.[5]

4. *Hybrid females*[6] *from rugose × glazed;* somatic appearance normal as in rugose females; many tested, all sterile. (See Metz and Bridges, loc. cit.)

5. *Hybrid females from rugose × wax;* somatic appearance normal as in rugose females; approximately 100 tested, one fertile, giving 19 normal females, 8 rugose males and 5 wax males.[7]

6. Hybrids of rugose, glazed or wax with any other mutants thus far used are all fertile.

Reviewing these results it is seen that in each hybrid involving rugose and either glazed or wax, rugose is dominant as regards somatic manifestation. But the sterility of the hybrids seems to indicate that the fertility

of rugose is recessive to the sterility of both glazed and wax. In the glazed-wax hybrid a similar relation holds for the somatic effect, glazed being dominant to wax. The high degree of sterility in both homozygous parent forms here, however, would naturally call for the observed sterility of the hybrid and eliminates the question of dominance with respect to this feature.

From the above standpoint sterility in the rugose-glazed and rugose wax hybrids is accounted for by assuming the dominance of sterility instead of assuming an incompatibility as was done in the previous paper when only the rugose-glazed hybrids were known. The present explanation calls for the occasional occurrence of fertile hybrids between rugose and glazed to correspond to the occasional fertility in pure glazed females. No such fertile hybrids have been obtained thus far, but this may be due merely to chance, since only a very few of them would be expected in the number of females tested. This conclusion is made still more probable by the occurrence of the one fertile hybrid between rugose and wax.

[1] These PROCEEDINGS, **3**, 1917 (673).

[2] These three characters are almost certainly allelomorphs, although the sterility of the hybrids and the dominance of the normal somatic constitution of rugose females makes it difficult to demonstrate the fact for the latter character. See Metz, C. W., *Genetics*, **3**, 1918 (116).

[3] The females were tested both singly and in mass cultures. The use of some mass cultures prevents recording the exact number of flies actually tested, for in these a few females may have died before they had a chance to breed.

[4] In each case these are such as to indicate fertility, not contamination or non-disjunction. In one case both parents were glazed, the offspring being 3 glazed females and 5 glazed males.

[5] It is difficult to obtain accurate determinations of the relative degrees of fertility of the males in these cases, for even the wax males breed fairly well, but in general the relations seem to be as described.

[6] Since the characters are sex linked and apparently allelomorphic no truly "hybrid" males can be obtained.

[7] In this and the preceding experiment offspring were obtained from two or three other bottles, but they were such as to indicate that they probably came from females heterozygous for one factor only, due to non-virginity or contamination.

STUDIES ON THE TRANSFORMATION OF THE INTESTINAL FLORA, WITH SPECIAL REFERENCE TO THE IMPLANTATION OF BACILLUS ACIDOPHILUS. I. FEEDING EXPERIMENTS WITH ALBINO RATS

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The type of bacteria developing in the alimentary tract may be influenced in a noteworthy manner, not only through special alterations

in the chemical composition of the diet, but also through the ingestion of living cultures of *Bacillus acidophilus* (Moro), with or without accompanying carbohydrates.

When added in sufficient amounts to the basal diet of bread and meat, namely 2 grams daily, lactose and dextrin caused a very pronounced transformation of the intestinal flora of white rats within three to six days, and stimulated the development of *B. acidophilus* to such an extent that it entirely dominated the flora, thus suppressing or displacing almost completely all other bacterial types commonly found in the enteric tract. The simple flora persisted as long as the feeding of these carbohydrates was continued. *B. bifidus* (Tissier) occasionally increased in numbers under these dietary conditions, but as a rule it remained relatively obscure.

Maltose, saccharose and glucose exercised no transforming influence on the types of bacteria present. These results are in perfect agreement with those of Distaso and Schiller (1914) and Hull and Rettger (1917). Torrey (1919) claims, however, to have obtained a moderate increase in the numbers of *B. acidophilus* as a result of sucrose feeding in dogs.

The most plausible explanation of the favorable influence of lactose and dextrin feeding on the implantation of *B. acidophilus* is one which rests to a large extent upon the fact that in the rat at least they are not completely absorbed before they reach the large intestine. Here they establish an optimum environment by serving as a readily available source of energy for *B. acidophilus*. This theory was advanced by Hull and Rettger who found a direct relationship between the rapid development of *B. acidophilus* and the presence of a reducing substance in the contents of the colon and rectum after lactose feeding. These findings have been confirmed in the present investigation, not only in connection with the administration of lactose but also dextrin. No reducing substances could be detected, however, after the feeding of maltose, sucrose or glucose.

The stimulating influence of lactose and dextrin on the development of aciduric organisms in the intestine does not depend upon increased acidity of the intestinal contents, as hydrogen ion concentration determinations made on the fecal material of the caecum and colon of rats receiving these carbohydrates showed. The hydrogen ion concentration limits remained essentially the same during the preliminary and the transforming periods of the different experiments.

The feeding of 1 gram of lactose or dextrin daily to white rats brought about a partial transformation which never amounted to more than 50% of the entire flora. These results are in harmony with the findings of Hull and Rettger (1917) who likewise demonstrated that 2 grams of lactose are necessary to cause a complete transformation. They showed

further that larger amounts than 2 grams of lactose stimulate *B. bifidus* to increased activity, and that when as much as 3 grams of lactose is administered daily the Tissier organism assumes the chief rôle, thus supplanting *B. acidophilus* even.

The addition of 1 cc. of *B. acidophilus* suspension (McFarland nephelometer turbidity scale of 5) to 1 gram of lactose or dextrin brought the *B. acidophilus* level to the same point as when 2 grams of either lactose or dextrin were fed. The simplification of flora was rapid and complete, and *B. acidophilus* persisted as the dominating type as long as the combined feeding of *B. acidophilus* and the carbohydrate continued. Furthermore, the administration of 2 cc. daily of the *B. acidophilus* culture or suspension without the carbohydrates results in the intestinal implantation of *B. acidophilus* and the complete suppression of the other bacterial types which are ordinarily present in the enteric tract of the white rat. The response of the different rats to the pure culture feeding was practically the same in all of the subjects. It was of special interest to note, further, that the various curves which represent the development of *B. acidophilus* in the feces of the rats were of practically the same height and character, whether 2 grams of lactose or dextrin, or 1 gram of either of these carbohydrates together with 1 cc. of the *B. acidophilus* culture, or 2 cc. of the culture alone were fed daily.

Post-mortem examination of the different sections of the alimentary canal of rats harboring the simplified flora in the lower intestine revealed a general distribution of *B. acidophilus* throughout the length of the intestine.

Since the bacteria within the digestive tract procure their pabulum directly or indirectly from the diet consumed by the host, it is but logical to assume that there must be a definite relationship between the chemical nature of the ingested food and the metabolic activities of the intestinal organisms. Many investigators have noted that marked changes in the intestinal flora follow sharp alterations in the diet, especially from a high protein to a high caloric diet and vice versa. However, the so-called "high calory diets" can at best accomplish a short-lived bacterial reformation only, that is a change in metabolic activities of organisms already present from the putrefactive to the fermentative type. This shifting of bacterial metabolism is not necessarily accompanied by an elimination of undesirable bacteria from the alimentary tract. Lactose and dextrin, among the carbohydrates studied, alone have the property of stimulating the development of aciduric organisms.

Repeated attempts in the present investigation to establish *B. bulgaricus* in the alimentary canal of albino rats through the administration of extremely large numbers (5 cc. of suspensions) of this organism were entirely unsuccessful. At no time was *B. bulgaricus* recovered from the

feces or from any portion of the digestive tract, whether the organism was fed alone or together with a utilizable carbohydrate. The contention of Metchnikoff and his followers that *B. bulgaricus* lends itself readily to implantation in the digestive tract of man receives no support whatever from the present observations upon white rats.

Although considerable doubt must be cast upon Metchnikoff's explanation of the merits of the sour-milk therapy, we may be compelled to accept the fundamental principle of his arguments. That is, bacterial implantation with the concomitant transformation of the intestinal flora, is without doubt, both desirable and possible, but it is possible only when living cultures of properly qualified organisms are employed, or when such specific diets are administered which will stimulate and favor such a transformation. *B. bulgaricus* is not of intestinal origin and is incapable of accommodating itself to intestinal conditions. *B. acidophilus*, on the other hand, is of intestinal origin and has shown itself to submit readily to implantation in the intestine of the white rat at least.

In many of the experiments of Metchnikoff and his pupils large quantities of milk were consumed as milk cultures of *B. bulgaricus*. Hull and Rettger showed that milk, when given in sufficient amounts to rats causes a rapid development of *B. acidophilus* in the intestine which may result in the complete suppression of other forms of bacteria. These observations were confirmed in the present investigations, not only for rats but also in numerous milk-feeding experiments on man. Milk owes this property to the lactose. It is not at all unlikely that what Metchnikoff and his pupils regarded as *B. bulgaricus* in the feces of animals and of human subjects was in reality *B. acidophilus*. These two organisms resemble each other so closely that one may easily be mistaken for the other. The above assumption is strengthened further by the statement of Belonowsky (1907) that the intestinal flora of nursing mice was of the same character as that of the older mice receiving the *B. bulgaricus* (milk) culture.

The feeding of *B. bulgaricus* without due regard to the use of milk can have little or no importance attached to it. The beneficial results which have been attributed to yogurt and other oriental sour-milk products have in all probability been due to the milk as such, rather than to the acid-producing bacteria contained in these products.

For a full account of the present investigation the reader is referred to the doctorate thesis of Harry A. Cheplin in the Yale University Library. The methods and results of these experiments, together with those conducted on man, will be published at an early date in unabridged form.

ON A SINGLE NUMERICAL INDEX OF THE AGE DISTRIBUTION OF A POPULATION[1]

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Read before the Academy, April 27, 1920

It is an obvious fact that the crude death rate of any community is influenced in a marked degree by the age distribution of the living population of that community. Before any critical conclusions about the true force of mortality can be drawn some sort of correction must be applied to take account of the age distribution of the population. If one desires to make any analysis of the correlation of death rates with each other or with sundry environmental factors it is an absolutely essential prerequisite that there be found some single numerical expression which will be an approximate index of the age distribution of the population of each locality dealt with.

From a mathematical viewpoint the problem presented is one incapable of *exact* solution. The problem mathematically is to find a single parameter which will fully describe so complex a curve as that of the age distribution of a population, the general form of which is indicated in figure 1. But this is impossible. No single parameter can possibly describe fully such a curve. The best that can be done is to approximate as closely as may be to the impossible ideal. It is the purpose of this paper to describe and illustrate an approximation which comes very close to the requirements, indeed quite sufficiently so for all practical statistical purposes in all cases where it has yet been tested out, and is easy to calculate.

In a recent paper,[2] I proposed as an index of differences in age composition of populations, and applied it to 40 American cities, the expression

$$\chi^2 = S\left(\frac{\Delta^2}{P}\right) \qquad (1)$$

"where Δ is the deviation for each of six age groups (viz, 0–4, 5–14, 15–24, 25–44, 45–64, 65 and over) of the percentage of the actual population of each city in 1910 in each age group, from the percentage in the same group in the Standard Population of Glover's[3] Life Table, denoted in the formula by P. S denotes summation of all six values. The value χ^2 measures the extent to which each city deviates in the age constitution of its population from a fixed standard, but does not tell the nature or kind of the deviation."

In the paper cited it was shown empirically that in spite of the obvious defect of this index mentioned in the last sentence, it was adequate for the restricted material and purposes there involved. Since that paper was published it has occurred to me that the outstanding defect of the former index will, to that degree of accuracy which is requisite for all practical

statistical purposes, be entirely removed by adopting as an age-constitution index the function

$$\phi = S\left\{\frac{\Delta^2}{P}\right\}(M - M_p) \qquad (2)$$

where S, Δ and P have the same significance as before, and M = mean age of living population in any community, M_p = mean age of persons in a stationary population unaffected by migration and which, assuming

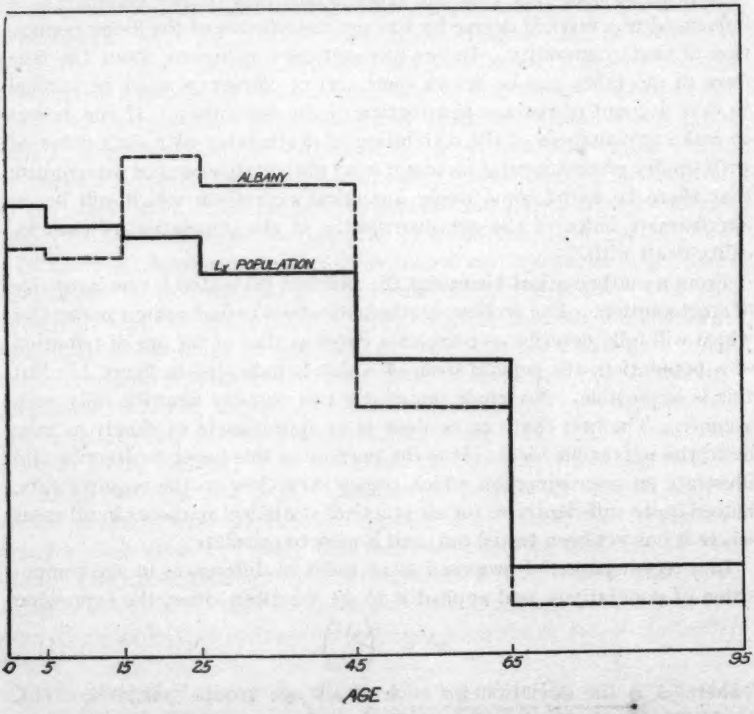

FIG. 1. Diagram showing age distribution of population, grouped in six age classes. The last class in this and other diagrams is assumed to have its ending at age 95. The solid line gives the stationary life table population, and the broken line the population of Albany. Total area under each curve = 100%.

the mortality rates of Glover's Life Table, would result if 100,000 persons were born alive uniformly throughout each year (M_p calculated from L_x line of Glover's Table (p. 16) = 33.796 years).

This procedure simply multiplies our former index χ^2 by the difference (given its proper sign) between the mean age of the observed population and the mean age of the standard population on the basis of which χ^2 was calculated. Since in fact the mean age of any actual urban population is never likely to be as great as the mean age of the stationary popula-

tion chosen as a standard of reference the actual values of ϕ will practically always be negative for cities. The smaller these negative values are numerically the greater will be the proportion of older persons in the population concerned. In short this function ϕ tells us not only the degree to which a given population deviates in its age distribution from a fixed standard age distribution, but also the nature of this deviation, whether on the one hand in the direction of a relative excess of aged, or

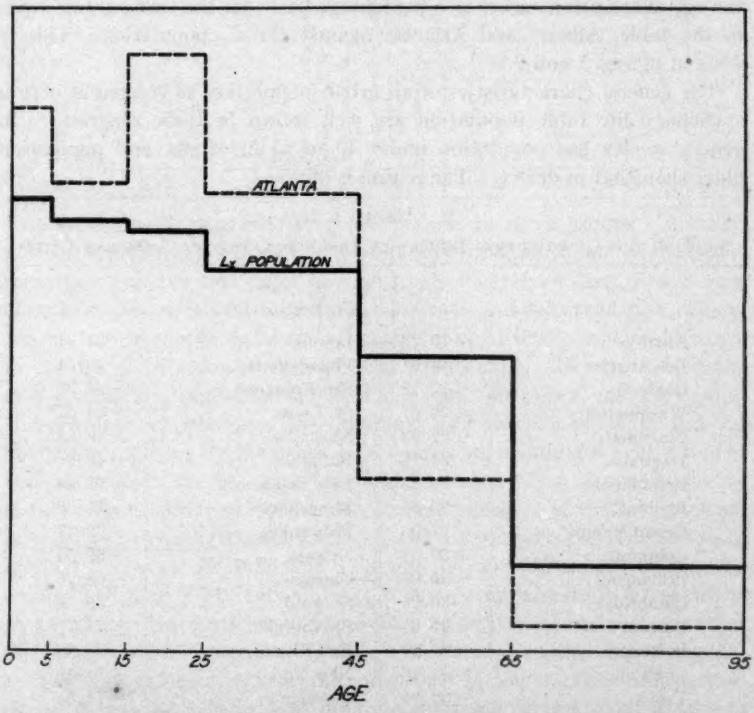

FIG. 2. Diagram showing age distribution of population, grouped in six age classes. The solid line gives the stationary life-table population, and the broken line the population of Atlanta. Total area under each curve = 100%.

on the other hand in the direction of a relative excess of the young. Theoretically it is possible for two populations differing from one another in a compensatory way to give the same values for the index ϕ. But two populations which differ in age distribution in any fundamental respect which could affect appreciably crude death rates will, in all populations I have been able to test, give different values of ϕ, provided the age classification from which the function is calculated is finely enough divided.

II

The numerical values of the index ϕ for a series of American cities, on the basis of the age distribution of the 1910 census are given by way of illustration in table 1. The cities are arranged in descending order of departure from the L_x population.

The extent of the departures from the standard population denoted by the several values of the indices may be indicated graphically by plotting the age distribution curve, on a percentage basis, for the two extreme cities in the table, Albany and Atlanta, against the L_x population. This is done in figures 1 and 2.

The general characteristics of an urban population as compared with a stationary life table population are well shown in these diagrams. In general a city has population under 40 to 45 in excess, and population older than that in defect. The reason is obvious.

TABLE 1

SHOWING AGE-CONSTITUTION INDICES OF THE POPULATION OF AMERICAN CITIES

CITY	ϕ	CITY	ϕ
Albany	− 10.73	Lowell	− 34.91
Los Angeles	− 13.61	Philadelphia	− 39.51
Oakland	− 17.18	San Francisco	− 43.35
Washington	− 20.67	St. Louis	− 44.57
Cincinnati	− 22.93	Nashville	− 48.32
Dayton	− 24.10	Buffalo	− 48.61
Indianapolis	− 24.23	New Orleans	− 52.44
Rochester	− 26.62	Minneapolis	− 55.39
Grand Rapids	− 27.95	Milwaukee	− 62.17
Columbus	− 29.08	Newark	− 62.54
Providence	− 30.05	Chicago	− 68.76
Cambridge	− 30.26	St. Paul	− 68.95
Boston	− 31.05	Pittsburgh	− 71.77
Baltimore	− 31.80	Fall River	− 73.39
Louisville	− 32.19	New York	− 74.42
Toledo	− 33.21	Cleveland	− 74.51
New Haven	− 33.70	Atlanta	− 82.71

III

It will be well to see in detail how the age distributions of some cities having nearly the same value for ϕ compare in respect of actual age distribution of their populations. Table 2 furnishes such comparisons for four pairs of cities, covering fairly the whole rank of values of ϕ shown in table 1.

From this table it is seen that as we pass from cities having a value of ϕ of about 24 to cities having ϕ equal to about 74 there is a steady change from populations having relatively many persons in the higher age groups

TABLE 2

PERCENTAGE AGE DISTRIBUTION OF THE POPULATION OF CERTAIN CITIES IN 1910

AGE CLASSES (YEARS)	DAYTON $\phi = -24.10$	INDIANAPOLIS $\phi = -24.23$	PROVIDENCE $\phi = -30.05$	CAMBRIDGE $\phi = -30.26$	NASHVILLE $\phi = -48.32$	BUFFALO $\phi = -48.61$	NEW YORK $\phi = -74.42$	CLEVELAND $\phi = -74.51$
0– 4	9.1%	8.0%	9.7%	10.3%	9.2%	10.0%	10.6%	11.1%
5–14	15.4	15.3	16.5	17.5	17.8	18.3	18.1	17.4
15–24	19.5	19.4	19.0	18.4	21.9	20.6	20.8	20.5
25–44	34.6	35.9	34.2	33.3	32.2	32.3	33.9	34.4
45–64	17.0	17.0	16.2	16.0	15.1	15.5	13.7	13.4
65 & over	4.4	4.3	4.2	4.4	3.8	3.4	2.8	3.0
Totals	100.0	99.9	99.8	99.9	100.0	100.1	99.9	99.8

to populations having relatively few persons in those groups. Furthermore, it is evident that in each of the four pairs of cities compared the agreement between the two cities having nearly identical values of ϕ is very close in respect of actual percentage distribution of the population. Dayton and Indianapolis were for all practical statistical purposes identical in the age distribution of their populations in 1910. So were Providence and Cambridge, Nashville and Buffalo, and New York and Cleveland. In each case the curve for one distribution of the pair winds in and out about the path set by the other. Of course, we should get finer differentiations brought out by the ϕ function if we used 15 or 20 age classes instead of the 6 here employed. But for the purposes of the investigation in which I am using these functions, and for purposes of illustration of method 6 classes are sufficient. A word of caution must, however, be emphasized here. The reliability of ϕ as an approximate index of differences in age distributions of population is greater as we pass in either direction towards the ends of its range of values. In the case of populations giving values of ϕ near the mean (say in the thirties for American cities) it may be necessary in order to get really differentiant values to calculate from a rather fine age grouping.

In general it is believed that the function of an age distribution of a population here proposed will give, in a single numerical expression, a substantially accurate indication of the essential nature of that age distribution, and will facilitate the differentiation and classification of populations in respect of this characteristic for statistical studies, particularly by the method of multiple correlation.

[1] Papers from the Department of Biometry and Vital Statistics, School of Hygiene and Public Health, Johns Hopkins University, No. 16.

[2] Pearl, R., "Influenza Studies. I—On Certain General Statistical Aspects of the 1918 Epidemic in American Cities," *Public Health Repts.*, **34**, 1919 (1743–1783).

[3] Glover, J. W., *United States Life Tables*, 1910, Bureau of the Census, 1916.

UPON AN IMPORTANT METHOD OF STUDYING PROBLEMS OF RELATIONSHIP AND OF GEOGRAPHICAL DISTRUBUTION

By Maynard M. Metcalf

The Orchard Laboratory, Oberlin, Ohio

Communicated by H. S. Jennings, May 28, 1920

In the course of the study of one hundred and thirty-four species and twenty subspecies of Opalinidae I have found data of great significance as to the geographical distribution and the former migrations of the Anura, and I wish to call attention to the importance of the method of studying such problems, in the case of each group of animals and plants, from the point of view both of the members of that group and of their parasites. A couple of illustrations will suffice to show how such study of a double series of data may present indisputable evidence as to the interpretation of the phenomena of present distribution of a family of animals.

The Leptodactylidae are the characteristic "frogs" of tropical and south-temperate America. They are found also in fair abundance in Australia and Tasmania and nowhere else. This has been taken to indicate former land connection between Patagonia and Australia by way of Antarctica. There are those, however, who have questioned this conclusion, believing the resemblance between these American and Australian "frogs" to be due to convergent or parallel evolution. But the latter hypothesis is definitely excluded by a study of the Opalinid parasites of the Leptodactylidae.

There is a genus of flat binucleated Opalinidae abundant in South America and Central America and spreading into the Antilles and a little into the southern part of North America. This genus, which I am naming *Zelleriella*, is the characteristic Opalinid parasite of the Leptodactylidae. The point of interest in the present connection is the fact that both American and Australian Leptodactylids have *Zelleriella* parasites, and the Australian Zelleriellas so closely resemble the American forms that it is difficult to separate them specifically. It might be conceivable that the Australian and American "frogs" assigned to the family Leptodactylidae may not be closely related and that their resemblances are due to parallel or convergent evolution, but it is not possible that both the Leptodactylids and their Opalinid parasites have evolved in parallel or convergent lines on the American and Australian continents. Such a coincidence is altogether improbable. *Zelleriella* is confined to Southern America and Australia.

Study of the Opalinid parasites of numerous families and subfamilies of Anura has given indication of a number of interesting things. It indi-

cates, for example, that at the time when Patagonia was connected with Antarctica there was, some bar to free migration between the northern part of South America and Patagonia, this bar being probably a shallow sea stretching across the middle of South America from ocean to ocean. Toads of the genus *Bufo* are abundant in tropical and temperate America, as they are in tropical and temperate Asia and the East Indies, but they have not reached Australia either by way of Antarctica or via New Guinea. The Bufos of South America carry *Zelleriella* parasites. Why did they not migrate to Australia by way of Antarctica carrying their Zelleriellas with them? The Leptodactylidae are a southern family having arisen apparently in the Patagonian region. The toads are a northern family. It seems clear that the toads were not present in Patagonia at the time when this region was connected with Australia by way of Antarctica. Apparently the shallow sea across the middle of South America separated the toads in the north from the Leptodactylids in the south. As Patagonia let go of Antarctica by subsidence of the intervening land area, there was a probably concomitant rise of land through what is now middle South America and the northern and southern portions of this continent came together. This allowed the toads and Leptodactylidae to meet, and the toads which are hospitable to all genera of Opalinidae accepted the Zelleriella parasites from the Leptodactylids. But it was now too late for the toads to migrate to Australia, the Antarctic route being barred by a wide stretch of newly formed ocean.

Numerous other problems of distribution are similarly illuminated by the study of the Anura and their Opalinid parasites. Other groups of Anuran parasites besides the Opalinidae should be studied with reference to problems of distribution, and every group of animals and of plants should similarly be studied with their parasites. In this way many questions of genetic relationship can be settled beyond doubt and many problems of geographical distribution and of migration routes can be solved. The method is one which should be used to the full because of the definiteness of the conclusions to which it leads. In all likelihood this method of study will in time settle the question of former land connection between Africa and northern South America as it has settled the question of former land connection between South America and Australia. The study of my own African and South American Opalinid material is not yet complete, so I do not discuss this problem here.

This paper is written merely to illustrate and emphasize the value of the method of studying relationships between groups of animals and plants, and their geographical distribution and migration routes, by means of comparison of their parasites and comparison of the distribution of the hosts with that of their parasites. The results in many cases are sure to be as surprisingly convincing as they have proven to be in the study of the Anura and their Opalinidae.

THE INFLUENCE OF COLD IN STIMULATING THE GROWTH OF PLANTS

By Frederick V. Coville

Smithsonian Institution, U. S. National Museum, Washington, D. C.

Communicated by C. D. Walcott. Read before the Academy, April 27, 1920

In regions having a cold winter like ours, with prolonged or repeated freezing, the native trees and shrubs, according to the general belief, become dormant because of the cold. It is also the general belief that warm weather is of itself the sufficient cause of the beginning of new growth in spring. Both these ideas are erroneous. The evidence now presented shows: first, that in our native trees and shrubs, dormancy sets in before cold weather, and that cold weather is not necessary for the establishment of complete dormancy; second, that after such dormancy has begun, the exposure of the plants to an ordinary growing temperature does not start them into growth; third, that these plants will not resume normal growth in the warm weather of spring unless they have been subjected previously to a period of chilling.

Dormant trees and shrubs which have had two or three months of chilling, either out of doors or in artificial cold storage, start into growth in the normal manner when spring comes, but if the dormant plants have been kept warm all winter they do not start into growth at the usual time in spring but continue their dormant condition for weeks or months, sometimes for a whole year, and when finally they do begin to grow their growth is of an abnormal character. One of these long dormant plants can readily be started into healthy growth, however, even after the expiration of a year, by subjecting it to a period of chilling. The best temperature for chilling is 32° to 40° Fahrenheit. It may be applied in either light or darkness.

In one of the greenhouses of the Department of Agriculture refrigerating machines have been installed in such a manner that plants supplied with normal daylight can be subjected at any period of the year to winter temperatures, even below freezing if desired.

By subjecting one part of a dormant bush to a chilling temperature and keeping the other part of it warm, the chilled portion can be brought into full leaf and flower while the other part remains completely dormant.

An important change takes place in the plant during the process of chilling. The starch stored in the cells is transformed to sugar, and not until this has been done can the plant utilize its store of starch in making its spring growth. Furthermore, the transformation of the starch into sugar creates high osmotic pressures within the plant. Certain sugar-exuding glands, called extra-floral nectaries, are interpreted as safety valves for the relief of excessive internal pressures which might burst the cells of the plant or otherwise derange its physiological activities.

The fact that our northern trees and shrubs after they become dormant in the fall require a period of chilling before warm weather will start them into growth again, is a protective adaptation of the highest importance to these plants, for if warmth alone would start them into growth they would begin growing in Indian summer and the stored food that the plant requires for its normal vigorous growth in the following spring would be wasted in a burst of new autumn growth, which would be killed by the first heavy freezes, and would be followed by a winter of weakness and probable death.

Further investigations on the effects of chilling are urged upon those engaged in experimentation bearing on the improvement of horticultural and agricultural practices. It is desirable especially to determine the proper temperatures for the storage of different kinds of seeds, bulbs, cuttings, and grafting wood; proper temperatures for the treatment of plants which are to be forced from dormancy to growth at unusual seasons; and proper temperatures for the storage of nursery stock, so that the nurseryman may have plants in proper condition for shipment on any date he desires.

ON THE NATURE OF THE NEGATIVE CARRIERS PRODUCED IN PURE HYDROGEN AND NITROGEN BY PHOTO-ELECTRONS

BY LEONARD B. LOEB*

RYERSON PHYSICAL LABORATORY, UNIVERSITY OF CHICAGO

Communicated by R. A. Millikan, June 5, 1920

As the result of an investigation of the cause of the abnormalities of the negative ions in air at low pressures (which is to appear shortly), it was found that the results obtained could be explained quite satisfactorily on the basis of a theory proposed by Sir J. J. Thomson.[1] This theory assumes that the electron does not attach to a neutral molecule to generate a negative ion on its first impact, but that on the average it will have a chance of uniting in one out of n impacts with a given type of molecule, where n is a constant which is a characteristic of the type of molecule considered. For air this constant was found to be in the neighborhood of 2.5×10^5. If the oxygen molecule is the molecule to which the negative electron attaches in air—a point of view for which there is considerable evidence—this means that in only one out of 5×10^4 encounters with oxygen molecules does the electron have a chance of attaching itself to a molecule to form a negative ion. It accordingly seemed of interest to see how nitrogen and hydrogen molecules behaved in respect to this theory, and to determine n for them if possible.

The method used for the investigation of this question was to measure the mobilities of the carriers generated as photoelectrons from one plate of a condenser using the Rutherford[2] alternating current method. Light from a quartz mercury arc passed through a quartz window sealed into the side of the brass case housing the condenser and was focussed upon the lower plate of the condenser by means of a quartz lens. The diameter of the plates was 10 cm. and the distance between them 1.65 cm. The upper plate of the condenser was connected to a quadrant electrometer

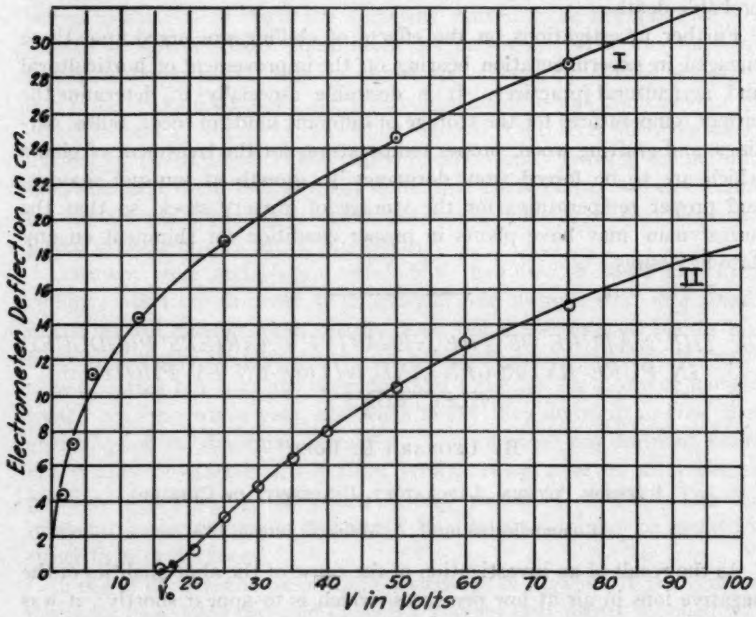

FIG. 1

Air. Press = 95 mm. I—Saturation; II—N = 148

of sensibility 3700 mm. per volt. The lower plate led to a commutator fed by a large battery of dry cells and capable of giving an alternating potential of square wave form of frequency from 15 cycles per second to 750 cycles per second. Connections were so arranged that the positive, or retarding, side of the wave was always about 20% higher than the corresponding negative, or accelerating, side of the wave.

The nitrogen gas which came from a tank of commercial nitrogen was passed over red hot CuO, red hot Cu, over KOH, and $CaCl_2$, over P_2O_5 and finally through a liquid air trap. The commercial hydrogen used was purified in the same way omitting naturally the tube of CuO. The system including the metal housing for the condenser plates was set up so as to eliminate as far as possible all organic vapors such as those coming

from stopcock greases and sealing waxes. The filling of the chamber was accomplished by exhausting to 1.5 cm. pressure and filling slowly with the purified gas six or seven times.

The curves numbered I in figures 1, 3 and 4, give the characteristic photoelectric saturation curves obtained in air at 95 mm. pressure, and in hydrogen and nitrogen at about 750 mm. pressure, when the current to the electrometer plate, under a *constant* accelerating potential is plotted against the potential. When the measurement of the current to the upper

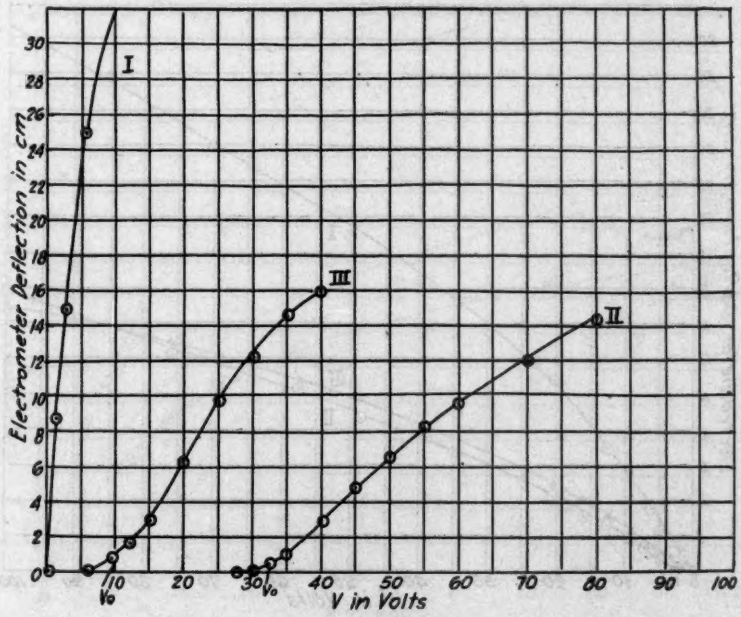

FIG. 2

H_2 [743 mm. O_2 12 mm. I—Saturation; II—Press = 293 mm., N = 360; III—Press = 151 mm., N = 364

plate was made using the alternating potentials the results obtained in air, and in hydrogen contaminated with O_2 are entirely different. The current remains nearly 0 until a certain critical voltage V_o is reached, and then rises rapidly eventually approximating a curve of the same shape as the saturation curve, but of half the ordinate. A curve of this type obtained in air at a pressure of 95 mm. using a frequency of alternation of 148 cycles is shown in figure 1, curve II. Figure 2, curves II and III show the results obtained in a mixture of 98.4% hydrogen and 1.6% oxygen at pressures of 293, and 151 mm., respectively. The frequency of the alternating potential was 362 cycles per second. Taking the value of the intercept of these curves with the voltage axis (V_o), one can de-

termine the velocity (u) of the ion in unit field from the equation $u = Nd^2/V_0$ (where d is the distance between the plates and N is the frequency of commutation). By multiplying u by the ratio $p/760$ (where p is the pressure at which the measurement was made) one obtains the mobility constant k of the ion. Now the mobility constant of an ion under normal conditions is of the order of magnitude of 2 cm. per second while that of the electron, though not definitely known, is of the order of magnitude of 200 cm. per second or greater. It is obvious then that by determining the

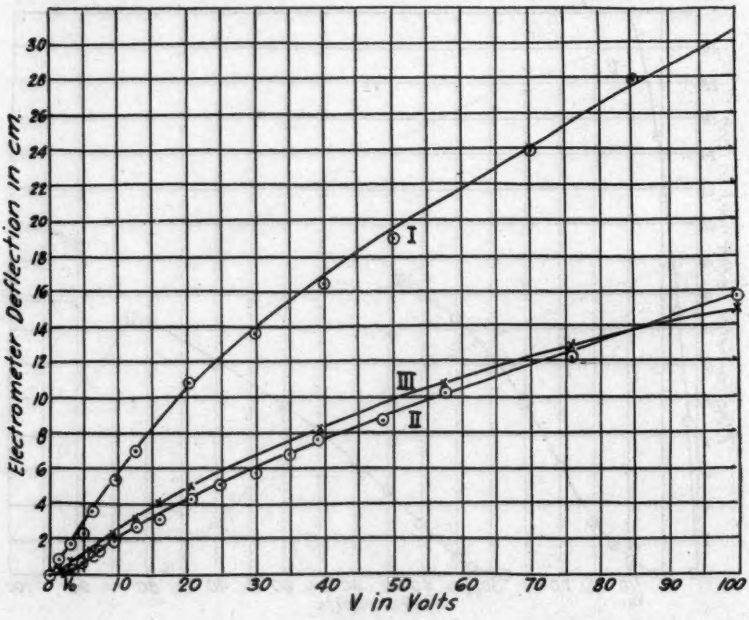

FIG. 3

Pure H_2. Press = 730 mm. I—Saturation; II—N = 712; III—N = 14.5

mobilities of the carriers in the gases the nature of the carriers may be determined.

The value of k obtained from the curves for air is 3.25 cm./sec. The values of k obtained in the mixture of hydrogen and oxygen above are 12.9 and 20.3† cm/sec. respectively, for the two pressures 293 and 151 mm. Since the normal values of k in air and in hydrogen are of the order of 2 cm./sec. and 7.5 cm. /sec. respectively, while those for the electrons in these gases are above 200 and 750 cm./sec., it is to be concluded that the carriers observed above are chiefly ionic in character.

When, however, these measurements were carried out in the carefully purified gases N_2 and H_2 the results were very different. These are illustrated in curves II and III, figure 4 for N_2, using frequencies of 714 and

35.7 cycles per second at a pressure of 753 mm. Curves II and III, figure 3, show the curves obtained in pure H_2 at a pressure of 730 mm. and using frequencies of 712 and 14.5 cycles, respectively.

These curves are practically saturation curves, their intercept with the voltage axis lying very close to the origin and being nearly independent of the frequency of alternation. Their ordinates are, however, but half of those for the corresponding saturation curves. This is to be expected since the time during which the electrometer deflection was measured

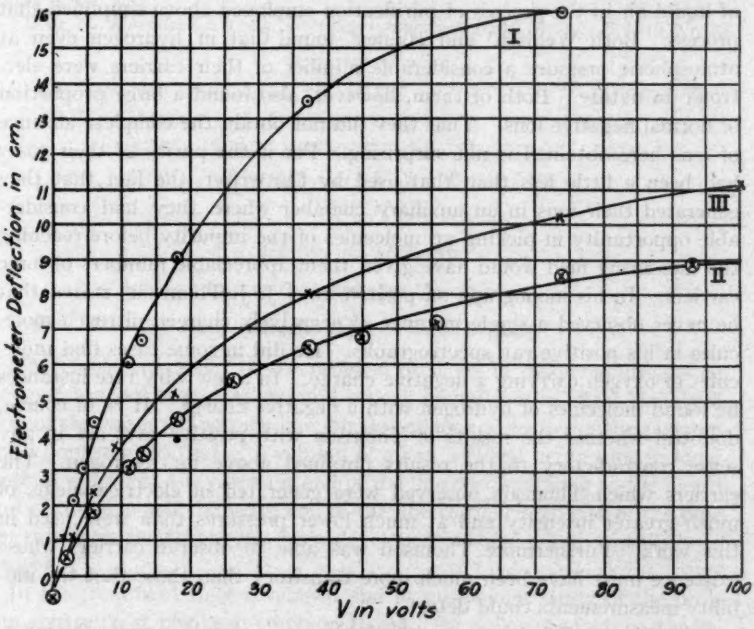

FIG. 4

Pure N_2. Press = 753 mm. I—Saturation; II—N = 714; III—N = 35.7

was the same for the saturation curves as for the curves taken with alternating potentials, while with the alternating potentials the accelerating field was on but half the time. The ordinates of the curves taken with the lower frequencies are in general slightly greater than those with the higher frequencies. This slight difference is probably due to the fact that the contacts at the commutator were slightly better at lower frequencies of alternation. The difference observed in the case of Nitrogen was due to a change in the intensity of the light from the mercury arc whose operating potential changed by two volts between the two determinations. Such a change easily accounts for the difference in the ordinates observed.

It is obvious that the carriers must be entirely electronic for the curves cut the voltage axis at values of V_0 so close to 0§ that the values of the

mobility computed therefrom are greater than 750 cm./sec. The conclusion to be drawn is that in pure nitrogen and hydrogen gas the electrons do not attach themselves to the molecules to form ions in any appreciable quantities.

The results obtained in nitrogen are in agreement with those of Franck.[3] The latter used a much more laborious method for the purification of his nitrogen than did the writer. He also used electrons generated by the action of alpha particles on the gas molecules. It is possible that the use of liquid air in the process of purification employed above simplified that process. Both Wellisch[4] and Haines[5] found that in hydrogen even at atmospheric pressure a considerable number of their carriers were electronic in nature. Both of them, however, also found a large proportion of normal negative ions. That they did not obtain the complete absence of ions here obtained is not surprising. For if the purity of their gases had been a little less than that used by the writer, the fact that they generated their ions in an auxiliary chamber where they had considerable opportunity in picking up molecules of the impurity before reaching the measuring field would have given them appreciable numbers of ionic carriers. In his monograph on positive rays J. J. Thomson[6] states that he never observed a single instance of negatively charged nitrogen molecules in his positive ray spectrographs. He did in some cases find molecules of oxygen carrying a negative charge. In a few vary rare instances he found molecules of hydrogen with a negative charge. It is, of course, doubtful whether the results of Thomson with positive rays are in any sense contradictory to the results obtained above for hydrogen. The carriers which Thomson observed were generated in electrical fields of much greater intensity and at much lower pressures than were used in this work. Furthermore Thomson was able to observe carriers whose existence must have been much more transitory than those that the mobility measurements could detect.

According to Bohr's[7] theory the hydrogen molecule permits the incorporation of a third electron into the system to form a stable negative hydrogen ion. It is interesting to note that this does not occur to any noticeable extent under the conditions of the writer's experiments.

In conclusion the writer wishes to thank Professor R. A. Millikan for his very kind advice and criticism.

* NATIONAL RESEARCH FELLOW

† The values of k obtained (p. 438) are somewhat greater than those given for normal ions. These abnormal values of the mobilities are the result of the low pressures used. Their explanation on the basis of the Thomson theory will constitute the body of a later paper.

§ It is impossible to locate the value of V_0 for the curves in Nitrogen. They are too small. The curve in hydrogen obtained with 712 cycles appears to cut the axis near 2 volts. The value of k for the hydrogen carrier deduced from this lies above 900.

However, too much faith cannot be placed on intercepts taken so near the origin, with small deflections.

[1] Thomson, J. J., *London Phil. Mag.*, Sept., 1915.

[2] Rutherford, E., *Proc. Cambridge Phil. Soc.*, **9**, 1898 (401).

[3] Franck, J., *Verh. Deuts. Phys. Ges.*, **12**, 1910 (613).

[4] Wellisch, E. M., *Phil. Mag.*, **34**, 1917 (199); also *New Haven, Amer. J. Sci.*, July, 1917.

[5] Haines, W. B., *Phil. Mag.*, Oct., 1915; also July, 1916.

[6] Thomson, J. J., *Rays of Positive Electricity*, Monographs on Physics, Longmans, Green & Co., 1913.

[7] Bohr, N., *Meddelanden fran k. Vetenskapsakademiens Nobel Institute*, **5**, 1919, No. 28.

SHOCK OR WATER RAM IN PIPE LINES WITH IMPERFECT REFLECTION AT THE DISCHARGE END AND INCLUDING THE EFFECTS OF FRICTION AND NON-UNIFORM CHANGE OF VALVE OPENING

BY W. F. DURAND

LELAND STANFORD JR. UNIVERSITY

Read before the Academy, April 27, 1920

The classic treatment of the problem of shock in pipe conduits, as developed by Joukovsky, Allievi and others, assimilates in effect the condition of the water in the pipe line during the manifestation of the phenomena in question to that of a column of air in a closed organ pipe in longitudinal vibration, the reservoir end of the line corresponding to the mouth end of the pipe and the valve end of the line to the closed end of the pipe. On this basis the theory has been developed in some detail, especially by Allievi.

In the treatment thus developed and in subsequent study of the problem generally, it has been common to omit the following factors, the existence of which must affect the result in actual cases:

(1) The influence of the velocity head $v^2/2g$.

(2) The influence of friction.

(3) The loss of energy through the discharge valve considered as a nozzle.

(4) The influence due to a time rate of valve area closure irregular, or other than uniform.

(5) The influence due to the fact that the valve end of the line is not closed completely so long as the valve is partly open (as in the *operation* of opening or closing). The analogy with the air in a closed organ pipe is, therefore, imperfect and in particular the reflection at this end must, under these conditions, be incomplete rather than complete, as commonly assumed.

The paper discusses the effect of the neglect of these conditions and a more general treatment, as an extension of the original treatment by Joukovsky, is then developed for taking care of conditions (1), (2), (3), (4).

Based on this treatment a further extension is then developed, including the influence due to imperfect reflection at the valve end.

Four different bases for the specification of the amount of reflection realized are presented and the treatment is developed to include any of these according to choice.

These four methods are as follows:

(1) A constant fraction or percentage of complete reflection.

(2) A fraction or percentage of complete reflection defined as $(a_1 - a)/a_1$ where a_1 = area of full valve opening and a = area of opening at any instant during valve movement.

(3) A fraction or percentage of complete reflection defined as $(A - a)/A$ where A = cross-section area of pipe and a = area of valve opening at any instant during valve movement.

(4) A fraction or percentage of complete reflection defined as $(v_1 - v)/v_1$ where for closure v_1 = full initial velocity and for opening v_1 = full final velocity, while v = velocity at any instant during valve movement.

The treatment throughout the paper includes both closure and opening of the valve, either in whole or in part, or in general any amount of valve movement either in closure or opening.

The paper is illustrated with diagrams showing typical numerical cases.

Of the various conditions thus brought within the scope of a form of analytical treatment, the author considers that part of the paper relating to No. 5 of more especial interest, representing, so far as he is aware, a distinct extension of earlier treatments of the problem and in such manner as to bring under some form of analytical control an important condition which must always prevail in actual cases of the development of shock or water ram in liquid conduits.

REPORT OF THE ANNUAL MEETING

PREPARED BY THE HOME SECRETARY

The annual meeting of the Academy was held in Washington, April 26, 27 and 28, 1920, at the Smithsonian Institution. The following eighty-eight members were present: C. G. ABBOT, BAILEY, A. GRAHAM BELL, BOAS, BOGERT, BRIDGMAN, BUMSTEAD, W. W. CAMPBELL, CARTY, CASTLE, CATTELL, F. W. CLARKE, J. M. CLARKE, CONKLIN, CREW, CROSS, CURTIS, DALL, DAVENPORT, DAVIS, DAY, DICKSON, DONALDSON, DUNN, DURAND, FEWKES, FORBES, FROST, GOMBERG, HALE, E. H. HALL, G. S. HALL, HALSTED, HARPER, HARRISON, HENDERSON, HILLEBRAND, HOLMES, HOWARD, HOWE, HOWELL, IDDINGS, JENNINGS, JEWETT, KAS,

NER, LANGMUIR, LEUSCHNER, LEWIS, LINDGREN, LUSK, LYMAN, MELTZER-MENDEL, C. E. MENDENHALL, C. H. MERRIAM, J. C. MERRIAM, MERRITT, MICHELSON, MILLIKAN, E. F. NICHOLS, A. A. NOYES, W. A. NOYES, PARKER, PEARL, RANSOME, REID, REMSEN, ROSA, RUSSELL, SCHLESINGER, SETCHELL, ERWIN F. SMITH, SQUIER, STIEGLITZ, STRATTON, THOMSON, THORNDIKE, TROWBRIDGE, VEBLEN, WALCOTT, WEBSTER, WELCH, WHEELER, D. WHITE, H. S. WHITE, EDWIN B. WILSON, R. W. WOOD, WOODWARD.

BUSINESS SESSIONS

The President announced the deaths of two members since the Autumn Meeting: LOUIS V. PIRSSON and HORATIO C. WOOD. The preparation of the biography of Louis V. Pirsson was assigned to Joseph P. Iddings.

The resignation of E. W. MORLEY as Chairman of the J. Lawrence Smith Fund was presented by the President, and accepted.

Sections and Committees.—The election of J. J. CARTY to succeed Gano Dunn as Chairman of the Engineering Section, term expiring in 1923, and of H. A. BUMSTEAD to succeed E. L. Nichols as Chairman of the Section of Physics, term expiring in 1923, were announced.

The President announced the following committee appointments:

COMMITTEE ON	APPOINTED	TO SUCCEED	TERM EXPIRING
Henry Draper Fund	H. N. Russell	Joseph S. Ames	1925
J. Lawrence Smith Fund	W. Cross, *Chairman*	E. W. Morley	1924
	Walter S. Adams	E. W. Morley	1924
	Frank Schlesinger	A. O. Leuschner	1925
Comstock Fund	John J. Carty	Elihu Thomson	1925
Marsh Fund	H. A. Brumstead	Edward S. Dana	1923
Murray Fund	Alfred G. Mayor	George H. Parker	1923
Marcellus Hartley Fund	David W. Taylor	Arthur G. Webster	1923
	William H. Welch	Theobald Smith	1923
Mary Clark Thompson	Waldemar Lindgren	Gano Dunn	1923

Reports.—The Annual Report of the President to Congress for 1919 was presented and distributed.

The report of the Treasurer was presented in printed form and approved.

The Home Secretary presented the following report:

THE PRESIDENT OF THE NATIONAL ACADEMY OF SCIENCES.

SIR:

I have the honor to present the following report on the publications and membership of the National Academy of Sciences for the year ending April 26, 1920.

Two parts of Volume 14 of the Memoirs of the National Academy of Sciences have been completed and distributed: the second memoir, "Complete Classification of Triad Systems," by H. S. White, F. N. Cole and L. D. Cummings, and the fourth memoir, "Minor Constituents of Meteorites," by G. P. Merrill.

The third memoir, "Tables of Minor Planets," by A. O. Leuschner, A. E. Glancy, and S. H. Levy, and the fifth and final memoir of Volume 14, "Tables of the Esponential Function," by C. E. Van Orstrand, are now in page proof and will be issued shortly, as will also Volume 15, "Psychological Examining in the United States Army," by Robert M. Yerkes.

Volume 16, first memoir, "Lower California and its Natural Resources," by E. W. Nelson, and the second memoir, "Studies upon the Life Cycles of Bacteria," by F. Löhnis, are now in galley proof. The third memoir, "A Recalculation of Atomic Weights," by F. W. Clarke, is now in the hands of the printer.

Volume VIII of the Biographical Memoirs has been completed with the publication of the biographies of Benjamin Osgood Peirce, and Cleveland Abbe, and the bound volume distributed. The following biographies forming a part of Volume IX have been completed and distributed: William Bullock Clark by John M. Clarke; Arnold Hague by Joseph P. Iddings; Eugene Waldemar Hilgard by Frederic Slate; James Dwight Dana by L. V. Pirsson; James Mason Crafts by Charles R. Cross; Lewis Boss by Benjamin Boss; and Alpheus Spring Packard by T. D. A. Cockrell. That of Charles Sedgwick Minot is now in page proof.

The Report of the National Academy of Sciences has been issued and the fourth Annual Report of the National Research Council will be issued in separate form in a few days. The Proceedings have reached the third number of the sixth volume.

Since the last meeting, two members have died. Louis V. Pirsson, elected 1913, died on December 8, 1919, and Horatio C. Wood, elected in 1879, died in 1919. This leaves an active membership of 175 members, 1 honorary member, and 31 foreign associates. Two foreign associates are reported dead; Gustav Retzius, elected in 1909, and Ivan Petrovich Pavlov, elected in 1908. The former died July 21, 1919.

(Signed) C. G. ABBOT, *Home Secretary.*

Reports from Trust Fund Committees.—The following telegraphic report from A. A. Noyes, Chairman of the Committee on the Barnard Medal, was read: Committee recommends awarding Barnard Medal to ALBERT EINSTEIN for highly original and fruitful development of the fundamental concepts of physics through application of mathematics.—A. A. NOYES.

The report of the Committee on the MURRAY FUND, singed by the Chairman, WM. H. DALL, was presented. The Committee recommended the award of the Agassiz medal to ADMIRAL C. D. SIGSBEE, U. S. N., retired, for his important contributions to oceanography, both by actual research, by publication of his results, and invention of new methods.

The Directors of the WOLCOTT GIBBS FUND reported the unexpended income of the fund as $965.56, of which $500.00 is at present invested in Liberty bonds.

The Trustees of the WATSON FUND recommended the following grants:

No. 20.—An appropriation of $500 to JOHN A. MILLER, Sproul Observatory, Swarthmore College, for the employment of assistance in measuring and reducing plates for the determination of parallaxes. This is a renewal of No. 17.

No. 21.—A further appropriation of $250 to J. A. PARKHURST for the employment of assistance in measuring and reducing plates for photographic and photovisual magnitudes in the north polar sequence and in the parallax field on which the Yerkes Observatory is working.

The cash balance of the Watson Fund on March 31, 1920, was $1,663.52, not including $1,000 accrued income invested in bonds.

The following report from H. F. OSBORN, Chairman of the ELLIOT FUND, was read:

The Committee is requesting recommendations for the award for the year 1919, the award to be made for the best zoölogical work of the year.

The Treasurer reports that on March 31, 1920, there was a cash balance of $564.81 in this fund, and that on the same date the sum of $300 of income was invested in Liberty Bonds.

The only appropriation from the GOULD FUND for the current year has been $600 to BENJAMIN BOSS for the purpose of meeting part of the expenses of the *Astronomical Journal*. The cash balance of this fund is $840.01; the invested income, $6,250.

The Committee on the HENRY DRAPER FUND report was presented. The Committee recommended the award of the DRAPER GOLD MEDAL to ALFRED FOWLER, F.R.S., Professor of Astrophysics, Imperial College, South Kensington, London, for contributions to our knowledge of sunspots, comets and the stars—especially red stars of Secchi's type III—on the basis of spectroscopic observations, and his interpretations of celestial phenomena.

The following grants were recommended:

$400 to S. A. MITCHELL, of the University of Virginia, to complete the purchase of a measuring microscope for use in the photographic determination of stellar parallaxes, on the basis of observations made with the 27-inch refracting telescope. The Academy awarded the sum of $250 from the Draper Fund to apply on the purchase of this instrument and the proposed grant of $400 will complete the purchase. The microscope, costing $650, becomes in effect the property of the Academy. Professor Mitchell will devote an equivalent sum, $400, to other needs of his parallax research.

$300 to JOEL STEBBINS, Professor of Astronomy in the University of Illinois, to assist in the further development of the photo-electric-cell photometer.

$400 to FRANK SCHLESINGER, Director of the Allegheny Observatory to enable him to test an automatic zenith camera for the determination of terrestrial latitudes, with the expectation that the results will be more accurate than any hitherto obtained by other means. It is proposed that this instrument be mounted temporarily at the International Latitude Observatory at Ukiah, California, where the astronomer in charge will operate it for a year or two as a labor of love. The grant is needed to install the instrument at Ukiah and to make certain auxiliary apparatus required in its operation. The Allegheny Observatory is loaning the objective and the photographic plates obtained will be measured by Dr. Schlesinger himself or under his immediate direction.

$175 to E. B. FROST, Director of Yerkes Observatory, for the purchase of a Hess-Ives tint photometer for use in the Yerkes Observatory, to supplement the Hartmann micrometer in the measurement of various illuminants, of the transmission of filters for various wave-lengths, of the absorption of photometric gratings, and of other phenomena and subjects.

$500 to Dr. ANTONIO ABETTI, Director of the Arcetri Observatory, Florence, Italy, to apply on the cost of a combined spectrograph and spectro-heliograph for use in combination with a 60-foot tower telescope now under construction. It is planned that this instrument shall be used by the son of the Director, Dr. Giorgio Abetti, well known to many American astronomers, recently transferred from the Observatory in Rome to the Arcetri Observatory.

$200 to MAJOR WILLIAM BOWIE, Chief of the Division of Geodesy, U. S. Coast and Geodetic Survey, in temporary support of the International Latitude Observatory at Ukiah, California, to assist in meeting an emergency due to the failure of the Observatory's regular source of funds.

The cash balance of the Draper Fund on March 31, 1920, was $1,065.04. Additional income to the amount of $917.50 is invested in bonds.

The following communications, presented by the Foreign Secretary, were referred to committees for further report:

1. A communication from the Académie des Sciences, Paris, signed by the President and the Perpetual Secretaries, requesting that the National Academy of Sciences use its influence to secure the adoption of the metric system of weights and measures in America.

2. A communication from Vienna, signed by the Actuary and other employees of the Akademie der Wissenschaften, requesting aid for themselves and their families in the way of food drafts, to relieve the situation now existent in Vienna because of the shortage of food.

3. A communication from the Oesterreichische Gesellschaft für Meteorologie, requesting funds for the support of its official publication, the "Zeitschrift für Meteorologie," and for the continuance of the work of its meteorological observatory. The total amount needed is 10,000 francs, or about $1,930.00.

Elections.—ARTHUR L. DAY and T. H. MORGAN were elected members of the COUNCIL to succeed E. G. Conklin and A. A. Noyes.

The following were elected to membership in the Academy:

JAMES ROWLAND ANGELL, Chairman of the National Research Council.
HENRY PRENTISS ARMSBY, State College, Pennsylvania.
WILDER DWIGHT BANCROFT, Cornell University.
HANS FREDERIK BLICHFELDT, Leland Stanford Jr. University.
ANTON JULIUS CARLSON, University of Chicago.
WILLIAM DUANE, Harvard University.
LEWIS RALPH JONES, University of Wisconsin.
ELMER PETER KOHLER, Harvard University.
CHARLES KENNETH LEITH, University of Wisconsin.
CLARENCE ERWIN MCCLUNG, University of Pennsylvania.
ELMER VERNER MCCOLLUM, Johns Hopkins University.
HARRIS JOSEPH RYAN, Leland Stanford Jr. University.
JOEL STEBBINS, University of Illinois.
BAILEY WILLIS, Leland Stanford Jr. University.

The following foreign associates were elected:

FRANK DAWSON ADAMS, President of McGill University.
MARIE ENNEMOND CAMILLE JORDAN, College of France.
FRANÇOIS ANTOINE ALFRED LACROIX, Musée d'Histoire Naturelle, Paris.
HEIKE KAMERLINGH ONNES, University, of Leyden.
SIR DAVID PRAIN, Royal Botanic Gardens, Kew, Surrey.
SANTIAGO RAMON Y CAJAL, University of Madrid.

Previous to adjournment, the following motion was adopted: That the Home Secretary be requested to transmit the thanks of the Academy

to the Smithsonian Institution and to the Cosmos Club for courtesies extended to the members of the National Academy of Sciences during the meeting.

SCIENTIFIC SESSIONS

A public lecture-discussion in the series of William ELLERY HALE LECTURES was delivered on *The Scale of the Universe*, on April 26 at 8.15 P.M., by HARLOW SHAPLEY, of the Mount Wilson Observatory and by HEBER D. CURTIS, of the Lick Observatory

Four public scientific sessions were held on Monday and Tuesday, April 26–27, at which the following papers were presented:

JOHN M. CLARKE: Conservation of natural resources as a proper function of the National Academy.

RAYMOND PEARL: On the rate of growth of the population of the United States since 1790 and its mathematical expression.

FRANZ BOAS: Growth and development as determined by environmental influences.

CHARLES B. DAVENPORT: Plural births in man.

SAMUEL J. MELTZER: The importance of the presence of both sympathetic superior cervical ganglia to the maintenance of life; and their possible relations to respiratory diseases.

CHARLES D. WALCOTT: Structure of Marrella and allied Middle Cambrian crustaceans.

JAMES R. ANGELL: The National Research Council.

ROBERT M. YERKES (introduced by E. L. Thorndike): A psychological study of the medical officers in the Army.

ROBERT W. WOOD: Spectroscopic phenomena of very long vacuum tubes.

L. T. E. THOMPSON, C. N. HICKMAN and N. RIFFOLT (introduced by Arthur G. Webster): The measurement of small time intervals and some applications, principally ballistic.

ROBERT A. MILLIKAN: The effect of molecular structure upon the reflection of molecules from the surface of liquids and solids.

ARTHUR G. WEBSTER: The Springfield rifle and the Leduc formula.

ARTHUR G. WEBSTER: Some new methods in internal ballistics of the Springfield rifle.

GEORGE E. HALE: The 100-inch Hooker telescope of the Mt. Wilson Observatory.

A. A. MICHELSON: (1) The vertical interferometer. (2) Preliminary tests in an attempt to measure the diameter of the stars. (3) A modification of the Foucault method adapted to long-distance measurement of the velocity of light.

ARTHUR G. WEBSTER: Preliminary measurements on the pressures in the "Onde de Choc."

ARTHUR G. WEBSTER: On the connection of the specific heats with the equation of state of a gas.

EDWIN H. HALL: Thermal conductivity of metals.

C. HART MERRIAM: Distribution and villages of the Indian tribes of the Klamath river region, California.

JOHN C. MERRIAM: Significance of correlation in function between the dentition and skeleton of the Sabre-tooth tiger.

GEORGE H. PARKER: On the colonial nervous system of Renilla.

DOUGLAS H. CAMPBELL: The genus Botrychium and its relationships.

FREDERICK V. COVILLE: The influence of cold in stimulating the growth of plants.

THOMAS B. OSBORNE and LAFAYETTE B. MENDEL: Some common foods as sources of vitamines.

LAWRENCE J. HENDERSON: The physico-chemical properties of haemoglobin.

WILLIAM A. NOYES and GEORGE H. COLEMAN: The direct combination of nitrogen and chlorine.

GILBERT N. LEWIS: Valence and chemical affinity.

WILLIAM F. DURAND: Shock or water ram in þipe lines with imperfect reflection at the discharge end and including the effects of friction and non-uniform change of valve opening.

LEONARD E. DICKSON: Recent notable progress in the theory of numbers.

EDWARD KASNER: Geodesics and relativity.

F. E. PERNOT (introduced by G. O. Squier): The use of alternating currrent for submarine cable transmission.

GEORGE O. SQUIER: Multiplex telephony and telegraphy over open-circuit bare wires laid in the earth or sea.

LYMAN J. BRIGGS: The air resistance of spheres.

ROBERT H. GODDARD: The possibilities of the rocket in weather forecasting.

H. FIELDING REID: The distribution of land and water on the earth.

WALDEMAR LINDGREN: The alterations of limestones in contact-metamorphism.

HENRY NORRIS RUSSELL: Notes on the internal constitution of the stars.

CHARLES GREELEY ABBOT: New observations on the variability of the sun.

RAYMOND PEARL: On a single numerical index of the age distribution of a population (by title).

EDWARD S. DANA: Biographical memoir of George Jarvis Brush (by title).

MABEL WEIL: Reports on the researches of the late Professor C. C. Trowbridge. (*a*) Auroras and magnetic disturbances. (*b*) Auroral phenomena and the meteor train zone. (*c*) An investigation of meteor trains. (*d*) Theories on the nature of meteor trains (by title).

THEODORE HOLM: The vegetation of the Alpine region of the Rocky Mountains in Colorado (by title).

S. W. WILLISTON: Osteology of the reptiles (by title).

IRVING LANGMUIR: The "quantel" theory: a new theory of the ether, matter and electro-magnetism.